Crouzeix's Conjecture

In 2004, Michel Crouzeix made the fascinating conjecture that for any n by n matrix A and any polynomial p

$$\|p(A)\| \leq 2\|p\|_{W(A)}. \tag{1}$$

Here $\|\cdot\|$ on the left-hand side denotes the operator 2-norm and $\|\cdot\|_{W(A)}$ on the right denotes the ∞-norm on the *field of values* or *numerical range*,

$$W(A) = \{q^*Aq : q \in \mathbb{C}^n, \quad q^*q = 1\}.$$

Crouzeix was able to prove the result with the constant 2 replaced by 11.08, but the method of proof could not be extended to give the conjectured value of 2 (which cannot be improved upon since there are matrices and polynomials for which $\|p(A)\| = 2\|p\|_{W(A)}$).

In this talk, I will discuss some background and related results, as well as applications. I will identify some classes of matrices / polynomials for which (1) is known to hold — e.g., 2 by 2 matrices and matrices whose field of values is a disk, linear polynomials and simple powers $p(z) = z^k$, $k = 1, 2, \ldots$ — and also some, seemingly trivial classes, for which it still is *not* known whether inequality (1) holds — e.g., general 3 by 3 matrices [Crouzeix recently proved the result for 3 by 3 nilpotent matrices], or general quadratic polynomials. I will discuss numerical studies and possible approaches to a proof.

This talk covers joint work with: Daeshik Choi, Vance Faber, Adrian Lewis, Michael Overton, and Nick Trefethen.