The Lanczos Algorithm in Finite Precision Arithmetic, Revisited

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Abstract

The Lanczos algorithm for computing eigenvalues/vectors of a real symmetric (or complex Hermitian) n by n matrix A uses a 3-term recurrence to construct orthonormal vectors from a sequence of Krylov subspaces span$\{q^1, Aq^1, A^2q^1, \ldots\}$. The eigenvalues of the tridiagonal matrix of recurrence coefficients, after some number $J < n$ steps, approximate some eigenvalues of A, and after n steps the eigenvalues of the tridiagonal matrix T_n are exactly those of A.

When the 3-term recurrence is perturbed slightly, as by rounding errors, although the individual perturbations are tiny, the effect on the results is dramatic. The constructed vectors are no longer orthogonal or even, necessarily, linearly independent. The tridiagonal matrix T_n may have several close approximations to some eigenvalues of A and no close approximations to others. YET, all is not lost. The eigenvalues of T_J still approximate some eigenvalues of A and, if run for sufficiently many steps, the method seems to find all eigenvalues, at the cost of generating multiple copies of some.

An explanation of this phenomenon was given many years ago [1, 2], when it was shown that a finite precision Lanczos computation for A behaves like the exact algorithm applied to a matrix $\hat{A}$ with many eigenvalues distributed throughout tiny intervals about those of A. The bound given on the size of these intervals was not as small as one might hope, however. In this talk, I will demonstrate how to construct such a matrix $\hat{A}$ and give some theoretical justification for the claim that the size of the intervals is on the order of $\epsilon\|A\|$, where ϵ is the machine precision.

References

[1] A. Greenbaum, *Behavior of Slightly Perturbed Lanczos and Conjugate Gradient Recurrences*, Lin. Alg. Appl. 113 (1989), pp. 7–63.

[2] A. Greenbaum and Z. Strakoš. *Behavior of Finite Precision Lanczos and Conjugate Gradient Computations*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 121–137.