On best approximation by polynomials of matrices

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Abstract

An important part of approximation theory is concerned with the approximation of a given function f on some (compact) set $\Omega \subset \mathbb{F}$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, by polynomials. Classical results in this area deal with the best approximation problem

$$\min_{p \in \mathcal{P}_k(\mathbb{F})} \|f - p\|_\Omega \qquad \text{where} \qquad \|g\|_\Omega = \max_{z \in \Omega} |g(z)|, \tag{1}$$

and $\mathcal{P}_k(\mathbb{F})$ is the set of polynomials of degree at most k with coefficients in $\mathbb{F}$.

Scalar approximation problems of the form (1) have been studied since the mid 1850s. Accordingly, numerous results on existence and uniqueness of the solution as well as estimates for the value of (1) are known. Here we consider a problem that at first sight looks similar, but apparently is much less understood: Let f be a function that is analytic in a neighbourhood of the spectrum of a given matrix $A \in \mathbb{F}^{n \times n}$, so that $f(A)$ is well defined, and let $\|\cdot\|$ be a given matrix norm. Consider the matrix approximation problem

$$\min_{p \in \mathcal{P}_k(\mathbb{F})} \|f(A) - p(A)\|. \tag{2}$$

In this presentation we will investigate two questions.

- If the norm is known to be strictly convex, as for example the Frobenius norm, then (2) is guaranteed to have a uniquely defined solution as long as the value of (2) is positive. A useful matrix norm is the matrix 2-norm (spectral norm), which for a given matrix A is equal to the largest singular value of A. This norm is not strictly convex, and thus the general result on uniqueness of best approximation in linear spaces with a strictly convex norm does not apply. We will show that the uniqueness result holds also for the matrix 2-norm.
- The optimality property of many useful methods of numerical linear algebra can be formulated as an approximation problem of the form

  $$\min_{p \in \mathcal{P}_k(\mathbb{F})} \|f(A)v - p(A)v\|, \tag{3}$$

  where $v \in \mathbb{F}^n$ is a given vector and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{F}^n$. If the given vector v has unit norm, which usually can be assumed without loss of generality, then an upper bound on (3) is given by (2). In order to analyse how close the upper bound (2) can possibly be to the quantity (3), one can maximize (3) over all unit norm vectors $v \in \mathbb{F}^n$ and investigate the sharpness of the inequality

  $$\max_{\substack{v \in \mathbb{F}^n \\ \|v\|=1}} \min_{p \in \mathcal{P}_k(\mathbb{F})} \|f(A)v - p(A)v\| \quad \leq \quad \min_{p \in \mathcal{P}_k(\mathbb{F})} \|f(A) - p(A)\|. \tag{4}$$

  From analyses of the GMRES method it is known that the inequality (4) can be strict. On the other hand, it is well known that if $A \in \mathbb{F}^{n \times n}$ is normal, then equality holds in (4). At least three different proofs of this result or variants of it can be found in the literature. In this presentation we will present yet another proof, which is rather simple because it fully exploits the link between matrix approximation problems for normal matrices and scalar approximation problems in the complex plane.

References

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