The computation of multiple roots of polynomials whose coefficients are inexact

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This lecture will show by example some of the problems that occur when the roots of a polynomial are computed using a standard polynomial root solver. In particular, polynomials of high degree with a large number of multiple roots will be considered, and it will be shown that even roundoff error due to floating point arithmetic, in the absence of data errors, is sufficient to cause totally incorrect results. Since data errors are usually larger than roundoff errors (and fundamentally different in character), the errors encountered with real world data are significant and emphasise the need for a computationally robust polynomial root solver.

The inability of commonly used polynomial root solvers to compute high degree multiple roots correctly requires investigation of the cause of this failure. This leads naturally to a discussion of a *structured condition number* of a root of a polynomial, where *structure* refers to the form of the perturbations that are applied to the coefficients. It will be shown that this structured condition number, where the perturbations are such that the multiplicities of the roots are preserved, differs significantly from the standard condition numbers, which refer to random (unstructured) perturbations of the coefficients. Several examples will be given and it will be shown that the condition number of a multiple root of a polynomial due to a random perturbation in the coefficients is large, but the structured condition number of the same root is small. This large difference is typically several orders of magnitude.

A method developed by Gauss for computing the roots of a polynomial will be discussed. This method has an elegant geometric interpretation in terms of pejorative manifolds, which were introduced by William Kahan (Berkeley). The method is rarely used now, but it will be considered because it differs significantly from all other methods (Newton-Raphson, Bairstow, Laguerre, etc.) and is non-iterative. The computational implementation of this method raises, however, some non-trivial issues – the determination of the rank of a matrix in a floating point environment and the quotient of two inexact polynomials – and they will be discussed because they are ill-posed operations. They must be implemented with care because simple methods will necessarily lead to incorrect results.

I will finish the talk by giving several non-trivial examples (polynomials of high degree, with several multiple roots of high degree, whose coefficients are corrupted by noise), and the results will be compared with other methods for the computation of multiple roots of polynomials whose coefficients are corrupted by noise.