Mathematical analysis of some fluid-structure interaction problems

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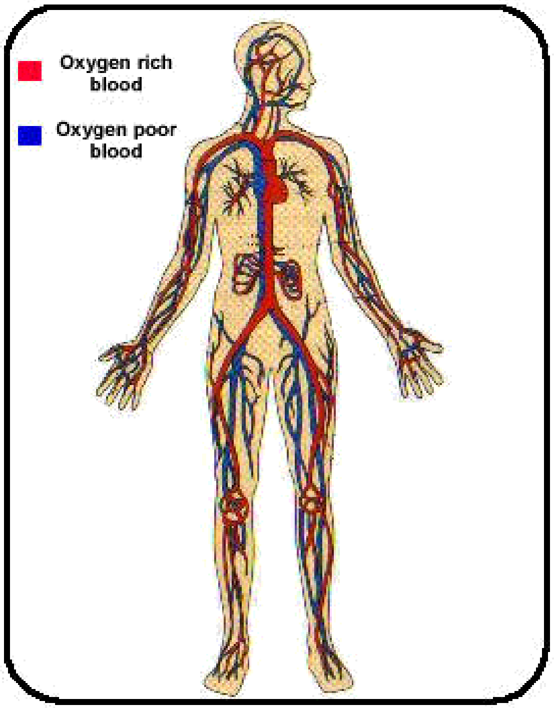
Outline of the lectures

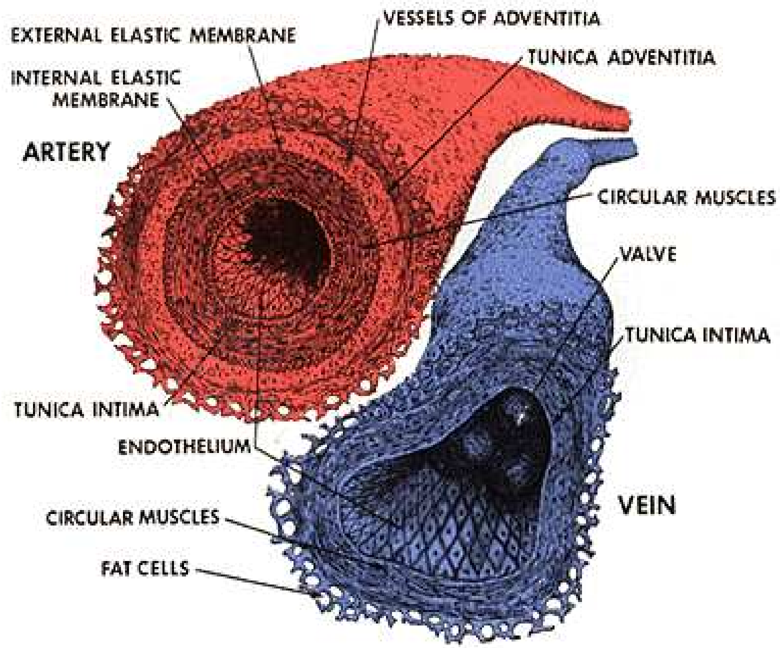
- Motivation.
- Introduction and some modeling aspects.
- Some mathematical preliminaries.
- A (short) survey of existence results.
- A result in the stationary case.
- Some results in the time-dependent case.
- A (very short) outline of numerical methods for FSI problems.

Motivation

Human circulatory system



Structure of large blood vessels



Introduction and some modeling aspects

Due to the highly complex structure of blood and arterial/venous vessels, one has to do drastical SIMPLIFICATIONS, i.e. one has to set up a MODEL.

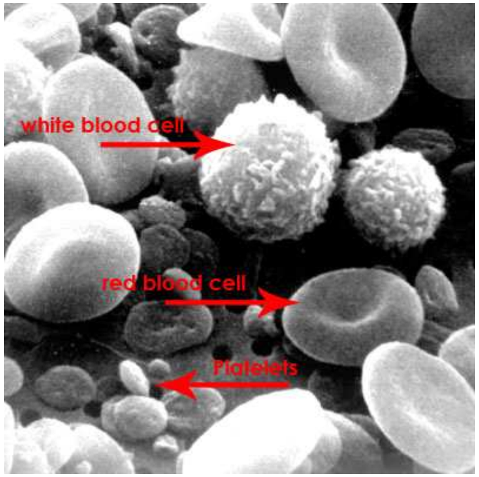
Modeling the circulatory system or even just a small part of it has to account for:

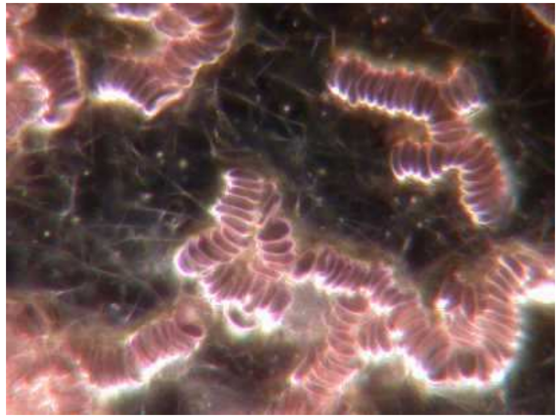
- the interdependence of blood and blood vessels: they are very complex, coexist and INTERACT
- the need of a model for blood (fluid)
- the need of a model for the arterial wall (elastic structure)
- the way to describe the interaction $\rightsquigarrow$ coupling
- the well-posedness of the model: does it make sense?
- the utility of the model: what does it tell us?

Introduction and some modeling aspects

How to model blood? Blood is a suspension of:

- plasma
- blood cells (erythrocytes, leucocytes, platelets)





Rheological properties of blood:

- in the absence of shear stress the erythrocytes form a continuous network
- increasing the shear stress up to the so-called yield stress breaks up the continuous structure of erythrocytes and allows the blood to flow.
- for a stress value above the yield stress erythrocytes tend to attach side by side to form rouleaux (aggregates).
- further increasing the shear stress breaks up the aggregates and finally only individual erythrocytes remain.

Introduction and some modeling aspects

More formally:

- blood is an incompressible fluid
- in large arteries it can be considered to be Newtonian $\rightsquigarrow$ (Navier-)Stokes equations
- actually, blood is a non-Newtonian fluid.

A Newtonian fluid is one for which the Cauchy stress tensor has the form $T_f = -p\mathbf{I} + 2\nu\mathbf{D}(\mathbf{v})$, where $\mathbf{D}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^t)$ is the stretching tensor. p denotes the pressure and $\mathbf{v}$ is the fluid's velocity.

Any other constitutive equation for the Cauchy stress tensor generates a non-Newtonian fluid model.

We will focus in these lectures on *Newtonian* fluids, however we also want to see some facts about the non-Newtonian ones, since they provide a more realistic (though still less encountered in fluid-structure interaction problems) modeling tool.

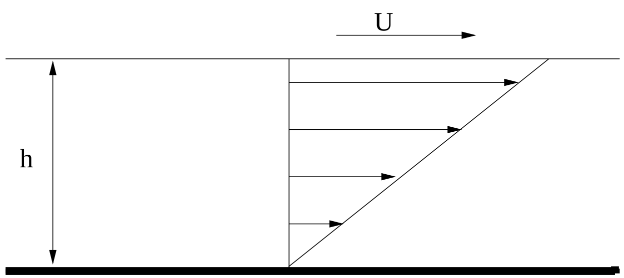
Generalized Newtonian fluids:

$$T_f = -p\mathbf{I} + \eta(\dot\gamma)\mathbf{D}(\mathbf{v})$$

with η a function of the principal invariants of the stretching tensor $\mathbf{D}$:

$$\begin{aligned} I_{\mathbf{D}} &= \text{trace } (\mathbf{D}(\mathbf{v})) = \text{div } \mathbf{v} = 0 \quad \text{(incompressibility)} \\ II_{\mathbf{D}} &= \frac{1}{2}[(\text{ trace } (\mathbf{D}(\mathbf{v})))^2 - \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{v})] = -\frac{1}{2}\mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{v}) \\ III_{\mathbf{D}} &= \det(\mathbf{D}(\mathbf{v})). \end{aligned}$$

$\eta(\dot\gamma)$ is also known as viscosity function and $\dot\gamma$ is the shear rate:



$$\begin{aligned} \dot\gamma &= U/h \quad \text{(rate of shear strain)} \\ \dot\gamma^2 &= -4II_{\mathbf{D}} = 2 \text{ trace } (\mathbf{D}(\mathbf{v}))^2 \end{aligned}$$

Remark: Newtonian fluids are particular cases of generalized (non-Newtonian) fluids, for $\eta = 2\nu = \text{constant}$.

Introduction and some modeling aspects

Some examples of generalized Newtonian fluids:

- Power-law fluids

$$T_f = -p\mathbf{I} + 2K|4II_{\mathbf{D}}|^{(n-1)/2}\mathbf{D}, \qquad \eta(\dot\gamma) = 2K\dot\gamma^{n-1},$$

  where n:=power-law index, K:=consistency.

  - for $n < 1$ (shear thinning):
    $\eta_0 := \lim_{\dot\gamma\to 0}\eta(\dot\gamma) = \infty$, $\eta_\infty := \lim_{\dot\gamma\to\infty}\eta(\dot\gamma) = 0$.
  - for $n > 1$ (shear thickening):
    $\eta_0 := \lim_{\dot\gamma\to 0}\eta(\dot\gamma) = 0$, $\eta_\infty := \lim_{\dot\gamma\to\infty}\eta(\dot\gamma) = \infty$.

- Powell-Eyring fluid (Powell, Eyring 1944)

$$\eta(\dot\gamma) = 2\Big(\eta_\infty + (\eta_0 - \eta_\infty)\frac{\sinh^{-1}(\dot\gamma\lambda)}{\dot\gamma\lambda}\Big), \qquad \eta_0,\ \eta_\infty,\ \lambda \quad \text{material constants.}$$

- Cross model (Cross 1965)

$$\eta(\dot\gamma) = 2\Big(\eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + (\dot\gamma\lambda)^{1-n}}\Big), \qquad \eta_0,\ \eta_\infty,\ \lambda,\ n \quad \text{material constants.}$$

Introduction and some modeling aspects

Viscoelastic fluids:

- Rivlin-Ericksen fluids:

  $T_f = -p\mathbf{I} + \zeta \mathbf{A}_1(\mathbf{v}) + \alpha_1 \mathbf{C}(\mathbf{v}) + \alpha_2 \mathbf{C}^2(\mathbf{v}) + \beta(\text{trace } (\mathbf{A}_1^2(\mathbf{v})) \cdot \mathbf{A}_1(\mathbf{v})$, where

  $$\begin{aligned} \mathbf{A}_1(\mathbf{v}) &= 2\mathbf{D}(\mathbf{v}) \quad \text{(Rivlin-Ericksen tensor)} \\ \mathbf{C}(\mathbf{v}) &= (\partial_t + \mathbf{v}\cdot\nabla)\mathbf{A}_1(\mathbf{v}) + \mathbf{A}_1(\mathbf{v})\nabla\mathbf{v} + (\nabla\mathbf{v})^t\mathbf{A}_1(\mathbf{v}). \end{aligned}$$

  - 3rd grade fluid: $\zeta \geq 0$, $\alpha_1 \geq 0$, $\beta \geq 0$, $-\sqrt{24\zeta\beta} \leq \alpha_1 + \alpha_2 \leq \sqrt{24\zeta\beta}$.
  - 2nd grade fluid: $\zeta \geq 0$, $\alpha_1 \geq 0$, $\beta = 0$, $\alpha_1 + \alpha_2 = 0$.
  - Newtonian fluid: $\beta = \alpha_1 = \alpha_2 = 0$.

- Oldroyd type fluids: $\mathbf{T}_f = -p\mathbf{I} + \tau$, $\tau = \tau_s + \tau_p$, where $\tau_s = 2\nu_s\mathbf{D}$ is the solvent contribution, and τ_p is the polymeric contribution.

- Generalized Oldroyd-B fluids: $\mathbf{T}_f = -p\mathbf{I} + \tau$, with

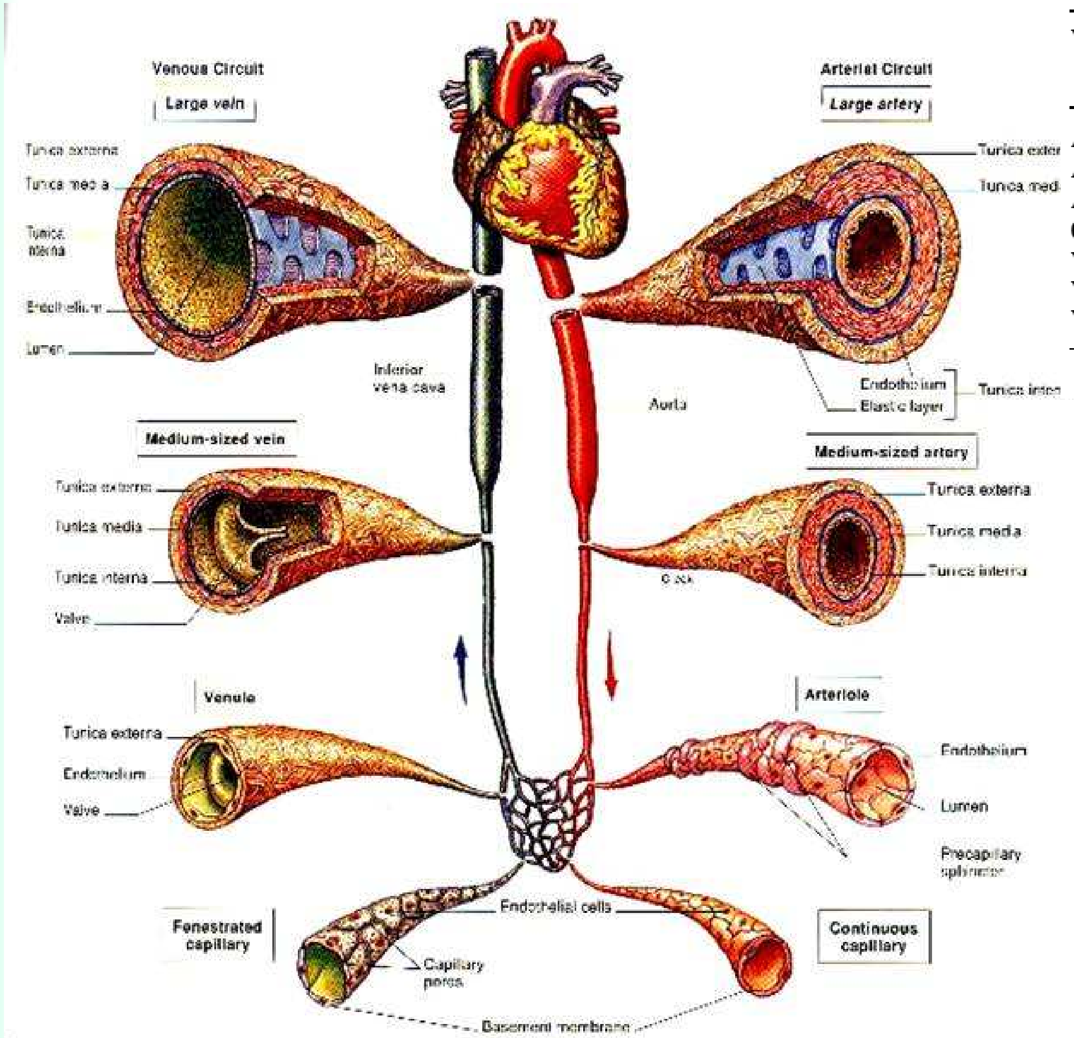
  $$\tau + \Lambda_1\Big[\dot{\tau} - (\nabla\mathbf{v})\tau - \tau(\nabla\mathbf{v})^t\Big] = \eta(\mathbf{A}_1)\mathbf{A}_1 + \Lambda_2\Big[\dot{\mathbf{A}}_1 - (\nabla\mathbf{v})\mathbf{A}_1 - \mathbf{A}_1(\nabla\mathbf{v})^t\Big],$$

  $\dot{\gamma} = \sqrt{\frac{1}{2}\text{trace } (\mathbf{A}_1^2)}$, thus $\eta(\mathbf{A}_1) = 2\Big(\eta_\infty + (\eta_0 - \eta_\infty)\frac{1+\ln(1+\Lambda\dot{\gamma})}{1+\Lambda\dot{\gamma}}$.

  Λ, Λ_1, Λ_2 are material constants, $\eta_0 = \lim_{\dot{\gamma}\to 0}\eta(\mathbf{A}_1)$, $\eta_\infty = \lim_{\dot{\gamma}\to\infty}\eta(\mathbf{A}_1)$.

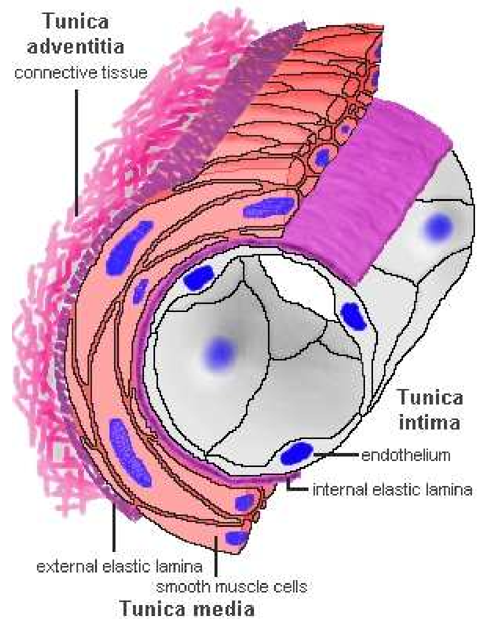
Modeling the blood vessels

blood vessels and their structure (http://www.biomed.metu.edu.tr/)



relationship between size, number of blood vessels and cross-sectional area (Quarteroni, Tuveri, Veneziani '00)

Vessel	Radius (cm)	Number	Area (cm^2)	Wall thickness (cm)
Aorta	1.25	1	4.5	0.2
Arteries	0.2	159	20	0.1
Arterioles	1.5×10^{-3}	5.7×10^{7}	400	2×10^{-3}
Capillaries	3×10^{-4}	1.6×10^{10}	4500	1×10^{-4}
Venules	1×10^{-3}	1.3×10^{9}	4000	2×10^{-4}
Veins	0.25	200	40	0.05
Vena cava	1.5	1	18	0.15



Introduction and some modeling aspects

Some models for blood vessels:

- membranar vascular walls (Navier equations) -neglect anisotropy and assume rotational symmetry (Renard '94):

$$\begin{aligned} \rho_w h \partial_{tt} u_r &= kGh\partial_{zz} u_r - \frac{Eh}{1-\zeta^2}\Big(\frac{\zeta}{R_0}\partial_z u_z + \frac{u_r}{R_0^2}\Big) + \Phi_1 \\ \rho_w h \partial_{tt} u_z &= \frac{Eh}{1-\zeta^2}\Big(\frac{\zeta}{R_0}\partial_z u_z + \partial_{zz} u_z\Big) + \Phi_2, \end{aligned}$$

  where $\mathbf{u}(r,z,t) = (u_r(r,z,t), u_z(r,z,t))$:= displacement of the wall,
  h:= wall thickness, $R_0(z)$:= vessel reference radius at rest,
  k:= Timoshenko shear correction factor, G:= shear modulus,
  ρ_w:= volumetric mass of the vessel wall,
  $\Phi_i, i = 1,2$ are forcing terms due to external forces.

- independent rings model (Perktold & Rappitsch '94) -neglect viscosity of the fluid and longitudinal displacements of the wall:

$$\rho_w \partial_{tt} u + \frac{E}{1-\zeta^2 R_0} u = \frac{1}{h}(p_w - p_0).$$

  Here $p_w - p_0$ is the transmural pressure: p_w:= pressure on the wall essentially due to the fluid, p_0:= reference value of the external pressure.

Introduction and some modeling aspects

Some models for blood vessels (continued):

- generalized string model (Quarteroni, Tuveri, Veneziani '00):

  $$\rho_w h \partial_{tt} u = a \partial_{zzzz} u + b \partial_{zz} u + c \partial_{tzz} u + du + \Phi.$$

  Here u is the radial displacement, a, b, c, d are constants depending on the material characteristics of the wall tissue, and Φ is the external force.

- more generally (comprising the 3D case): $\rho_w \partial_{tt} \mathbf{u} = \mathrm{Div}\, \mathbf{T}_w + \rho_w \mathbf{f}_w$, where $\mathbf{T}_w$ is the Piola-Kirchhoff stress tensor for the arterial wall. We shall come back later to this situation.

- circular cylindrical shell model (only radial displacements):

  $$\partial_{tt} u_r + u_r + k(\partial_{zzzz} u_r + 2\partial_{zz\theta\theta} u_r + \partial_{\theta\theta\theta\theta} u_r) = f_{w,rad} + f_{f,rad},$$

  where $f_{w,rad}$:= the radial component of the force applied to the elastic wall, $f_{f,rad}$:= the radial component of the force applied by the fluid, $\theta \in [0, 2\pi)$:= the angular coordinate, $k = \frac{K}{DR_0^2}$ is a dimensionless parameter, whose value decreases with decreasing thickness of the shell, $K = \frac{Eh^3}{12(1-\zeta^2)}$:= flexural stiffness, $D = \frac{Eh}{1-\zeta^2}$:= extensional stiffness.

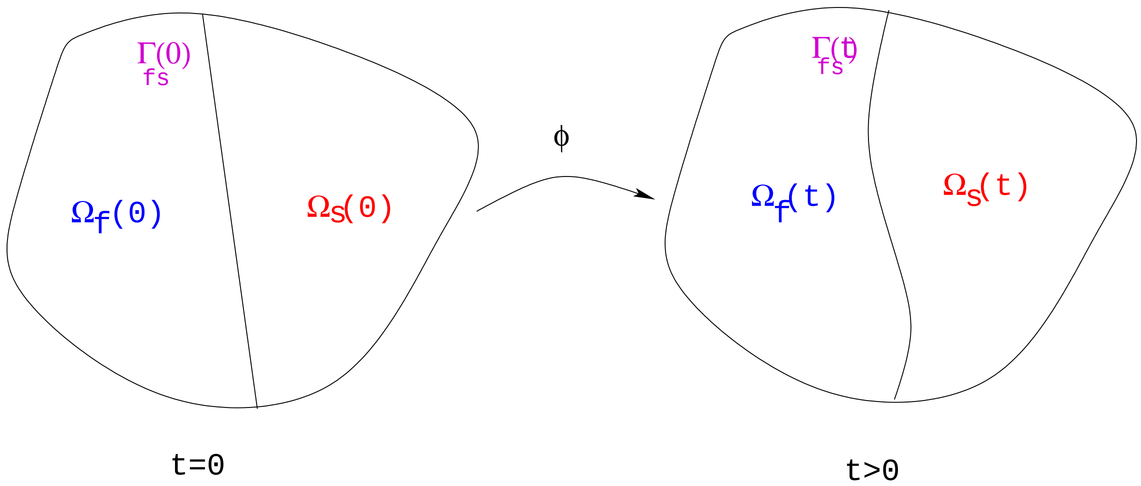
Introduction and some modeling aspects

Basic features: a fluid is moving in contact with an elastic structure. They interact at the interface between the two media (fluid/solid).

Basic challenges:

- the fluid-structure interface depends on time
- the fluid equations are naturally given in the Eulerian description, while those for the elastic structure follow a Lagrangean one. One typically carries out an Euler-to-Lagrange transformation of the fluid formulation.
  - Advantages:
    - a unified description
    - time-independent interface
  - Drawback: more complicated equations for the fluid.
- one tipically deals with parabolic equations for the fluid and with hyperbolic equations for the structure. Since their solutions have different regularity properties, this leads to significant difficulties when coupling the two media. However, these difficulties disappear:
  - in the stationary case, since both systems become elliptic
  - if one adds a hyperviscosity term to the structure equations
- the coupled problem is highly nonlinear.

Introduction and some modeling aspects



Notations:

Ω_s: domain of the elastic structure; Ω_f: domain of the fluid;

$\Omega(0) = \Omega_s(0) \cup \Omega_f(0)$ (reference configuration);

$\Omega(t) = \Omega_s(t) \cup \Omega_f(t)$ (current configuration);

$\Gamma_{fs}(0)$: fluid-structure interface in the reference configuration;

$\Gamma_{fs}(t)$: fluid-structure interface in the current configuration;

$\mathbf{X} \in \Omega(0)$: material coordinates;

$\mathbf{x} \in \Omega(t)$: coordinates in the current configuration;

$\mathbf{u}(\mathbf{X}, t)$: displacement of the elastic structure. Thus, $\phi(\mathbf{X}, t) = \mathbf{u}(\mathbf{X}, t) + \mathbf{X}$ is the deformation of the elastic structure.

$\mathbf{v}(\mathbf{x}, t)$: velocity of the fluid; $p(\mathbf{x}, t)$: fluid pressure.

Introduction and some modeling aspects

Basic equations of motion:

- For the (viscous, incompressible) fluid:

$$\begin{aligned} \rho_f(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) &= \text{div}\,\mathbf{T_f} + \rho_f \mathbf{f}_f \quad \text{in} \quad \Omega_f(t),\ t\in(0,T) \\ \text{div}\,\mathbf{v} &= 0 \quad \text{in} \quad \Omega_f(t),\ t\in(0,T) \\ &+ \quad \text{boundary conditions} \\ &+ \quad \text{initial conditions}, \end{aligned}$$

  where $\mathbf{T_f}$ is the fluid Cauchy tensor and ρ_f is the fluid's density.

- For the elastic structure:

$$\begin{aligned} \rho_s \partial_{tt}\mathbf{u} &= \text{Div}\,\mathbf{T}_s + \rho_s \mathbf{f}_s \quad \text{in} \quad \Omega_s(t),\ t\in(0,T) \\ &+ \quad \text{boundary conditions} \\ &+ \quad \text{initial conditions}, \end{aligned}$$

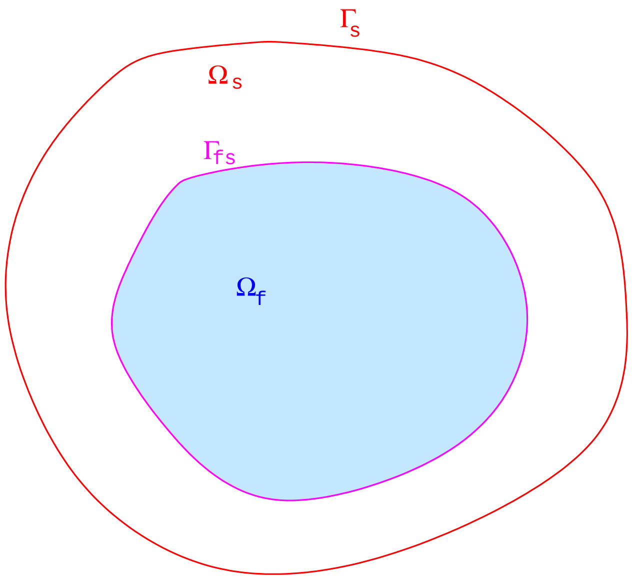
  with Div $:=$ div $_{\mathbf{X}}$, $\mathbf{T_s}$ the Piola-Kirchhoff stress tensor, and ρ_s the density of the elastic structure.

- For the coupling: continuity of velocities and of the stress distribution on $\Gamma_{fs}(t)$.

Which boundary conditions should we choose?

This is of course depending on the specific problem we want to consider:

Example 1: fluid completely enclosed by a thick elastic vessel



- BCs for the fluid: here the interface Γ_{fs} coincides with the fluid boundary, thus we ask for continuity of velocities on Γ_{fs}: $\mathbf{v}(\mathbf{x}) = \partial_t \mathbf{u}(\mathbf{X})$.
- BCs for the elastic structure: $\mathbf{u} = 0$ on Γ_s (the wall is clamped on its outer boundary) $\mathbf{T}_f(\mathbf{v}, p) \cdot n = \mathbf{T}_s(\mathbf{u}) \cdot n$ on $\partial\Omega_s \setminus \Gamma_s$. Actually, here the interface Γ_{fs} coincides with the inner boundary of the elastic vessel, i.e. $\Gamma_{fs} \equiv \partial\Omega_s \setminus \Gamma_s$. Also notice that we have different settings for the fluid and the solid equations (Euler vs. Lagrange).

Introduction and some modeling aspects

Example 2: fluid moving in a channel bounded by a flexible structure

- BCs for the fluid:
  - at the ends of the channel: periodicity, prescribed velocity (direction), prescribed pressure, 'do nothing', a.o.
  - on the rest of the boundary (fluid-structure interface Γ_{fs}): equality of velocities and of stresses
- BCs for the elastic structure:
  - on the outer boundary: $\mathbf{u} = 0$ (if Γ_s is clamped), respectively $\mathbf{n} \cdot \mathbf{T}_s(\mathbf{u}) = 0$ (if $\Gamma_s(t)$ is allowed to move)
  - on the inner boundary (fluid-structure interface Γ_{fs}): equality of velocities and of the stresses
  - at the ends of the channel: periodicity, clamped, ???.
    Regularity problems arise when two different types of BCs (e.g., Neumann and Dirichlet) meet on a portion of the boundary (Ciarlet '88).

Introduction and some modeling aspects

The analysis of the previous models can become very complicated. One can make certain simplifications:

Isotropic elasticity (same response of the material in all directions):

The most general constitutive equation is

$$\mathbf{T}_s = \phi_0 \mathbf{I} + \phi_1 \mathbf{B} + \phi_2 \mathbf{B}^2,$$

where $\mathbf{B}$ is the left Cauchy-Green strain tensor. The Lagrangean displacement is

$$\mathbf{u}(\mathbf{X}, t) = \phi(\mathbf{X}, t) - \mathbf{X},$$

and the Eulerian displacement is

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \phi^{-1}(\mathbf{X}, t).$$

Then $\mathbf{B} = \mathbf{F}^t \mathbf{F}$, where $F_{ij} = \frac{\partial \phi_i}{\partial X_j}$ is the deformation gradient.

$\mathbf{C} = \mathbf{F}\mathbf{F}^t$ is the right Cauchy-Green tensor.

The tensors $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$ are called the Lagrangean, respectively the Eulerian strain tensor.

Introduction and some modeling aspects

The Lagrangean gradient is Grad $\mathbf{u} = \mathbf{F} - \mathbf{I}$, thus $\mathbf{C} = (\mathbf{I} + \text{Grad } \mathbf{u})(\mathbf{I} + (\text{Grad } \mathbf{u})^t)$ and

$$\mathbf{E} = \frac{1}{2}[\text{Grad } \mathbf{u} + (\text{Grad } \mathbf{u})^t + (\text{Grad } \mathbf{u}) \cdot (\text{Grad } \mathbf{u})^t].$$

Observe that $F_{ij}^{-1} = \frac{\partial \phi_j^{-1}}{\partial x_i}$, s.t. the Eulerian gradient writes grad $\mathbf{u} = \mathbf{I} - \mathbf{F}^{-1}$, therefore $\mathbf{B}^{-1} = (\mathbf{I} - \text{grad } \mathbf{u})(\mathbf{I} - (\text{grad } \mathbf{u})^t)$ and it follows that

$$\mathbf{e} = \frac{1}{2}[\text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^t + (\text{grad } \mathbf{u}) \cdot (\text{grad } \mathbf{u})^t].$$

Small displacement gradients:

If we have to do with a material which does not infer substantial deformations, we can assume that the spatial rate of change of the displacement is very small, i.e.

$$\text{Grad } \mathbf{u}(\mathbf{X}, t) = \mathcal{O}(\epsilon), \qquad \epsilon \ll 1, \quad \forall\, \mathbf{X},\ t.$$

Neglecting all terms of order $\mathcal{O}(\epsilon^2)$ we obtain that the Lagrangean strain tensor $\mathbf{E}$ is aproximated by the infinitesimal Lagrangean strain tensor

$$\mathbf{E}_0 = \frac{1}{2}(\text{Grad } \mathbf{u} + (\text{Grad } \mathbf{u})^t).$$

Introduction and some modeling aspects

Observe that the Lagrangean and Eulerian gradients of $\mathbf{u}$ coincide at first order in ϵ:

$$\text{grad } \mathbf{u} = \text{Grad } \mathbf{u} + \mathcal{O}(\epsilon^2).$$

Indeed, we can write

$$\begin{aligned}
\mathbf{I} - \mathbf{F}^{-1} &= \mathbf{I} - [\mathbf{I} + (\mathbf{F} - \mathbf{I})]^{-1} = \mathbf{I} - [\mathbf{I} - (\mathbf{F} - \mathbf{I}) + (\mathbf{F} - \mathbf{I})^2 + \dots] \\
&= \mathbf{F} - \mathbf{I} + \mathcal{O}((\mathbf{F} - \mathbf{I})^2) \\
&= \mathbf{F} - \mathbf{I} + \mathcal{O}((\text{Grad } \mathbf{u})^2) \\
&= \mathbf{F} - \mathbf{I} + \mathcal{O}(\epsilon^2).
\end{aligned}$$

The statement follows from $\text{Grad } \mathbf{u} = \mathbf{F} - \mathbf{I}$ and $\text{grad } \mathbf{u} = \mathbf{I} - \mathbf{F}^{-1}$.

As a consequence, we can use the Eulerian formulation for both the fluid and the structure equations. Moreover, observe that the nonlinear strain tensors $\mathbf{E}$ and $\mathbf{e}$ have become linear.

This makes life easier, especially when studying the stationary case. However, in spite of the above simplification, the coupled problem (with the corresponding boundary and initial conditions) is still very difficult to solve in the time-dependent case, mainly because of the time moving interface $\Gamma_{fs}(t)$, which is itself unknown.

Introduction and some modeling aspects

Actually, $\Gamma_{fs}(t) = \{\phi \in \mathbb{R}^3 \ : \ \phi(\mathbf{X},t) = \mathbf{u}(\mathbf{X},t) + \mathbf{X}, \quad \mathbf{X} \in \Gamma_{fs}(0)\}$.

Time-independent interface:

Assume $\mathbf{u}$ is infinitesimal and time-independent on the interface. Then $\Gamma_{fs}(t) = \Gamma_{fs}(0)$, $\forall\, t \geq 0$. This is a substantial simplification, which allows handling the coupled system in an easier way, however the price we pay is the lost of an important feature of our problem.

The fluid-structure interaction problem becomes

$$
\begin{aligned}
\rho_f(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) &= \operatorname{div} \mathbf{T_f} + \rho_f \mathbf{f}_f \quad \text{in} \quad \Omega_f(0)\times(0,T) \\
\operatorname{div} \mathbf{v} &= 0 \quad \text{in} \quad \Omega_f(0)\times(0,T) \\
\mathbf{v}(0) &= \mathbf{v}_0 \quad \text{in} \quad \Omega_f(0) \\
\rho_s \partial_{tt}\mathbf{u} &= \operatorname{div} \mathbf{T}_s + \rho_s \mathbf{f}_s \quad \text{in} \quad \Omega_s(0)\times(0,T) \\
\mathbf{u} &= 0 \quad \text{on} \quad \Gamma_s \times (0,T) \\
\mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in} \quad \Omega_s(0) \\
\mathbf{v} &= \partial_t \mathbf{u} \quad \text{on} \quad \Gamma_{fs}(0)\times(0,T) \\
\mathbf{T}_f\cdot\mathbf{n} &= \mathbf{T}_s\cdot\mathbf{n} \quad \text{on} \quad \Gamma_{fs}(0)\times(0,T)
\end{aligned}
$$

Some mathematical preliminaries

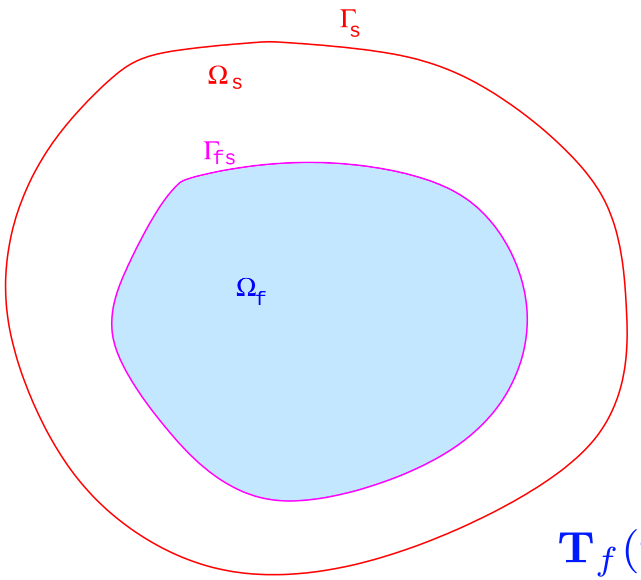
Question: is the involved coupled initial boundary value problem well posed?

Answers:

- Who cares?
  Studying well posedness can help:
  - to better understand the numerical issues arising at the discretization level
  - getting some clues about how to do the numerics
  - modifying the model in a reasonable way.
- In which sense?
  There are some nonexhaustive existence results. One always has to specify in which sense is meant the solution.
- Maybe. Apparently yes.
  One can build very complex models for which it is very hard to prove existence, though numerics seems to do well.

Some mathematical preliminaries

Once we have set up our model, energy estimates can tell us a great deal about it. To illustrate this, let us consider a typical situation:



- The fluid equations:

$$\begin{aligned} \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \text{div}\,\mathbf{T_f} &= \mathbf{f}_f \quad \text{in } \Omega_f(t) \\ \text{div}\,\mathbf{v} &= 0 \quad \text{in } \Omega_f(t) \\ \mathbf{v}(0) &= \mathbf{v}_0 \quad \text{in } \Omega_f(0) \end{aligned}$$

- The coupling:

$$\begin{aligned} \mathbf{v}(\mathbf{x}+\mathbf{u}(\mathbf{x},t),t) &= \partial_t\mathbf{u}(\mathbf{x},t), \quad \mathbf{x}\in\Gamma_{fs}(0) \\ \mathbf{T}_f(\mathbf{v},p)(\mathbf{X}+\mathbf{u}(\mathbf{X},t),t)\cdot\mathbf{n} &= \mathbf{T}_s(\mathbf{u}(\mathbf{X},t))\cdot\mathbf{n}(\mathbf{X},t) \end{aligned}$$

- The equations for the elastic structure:

$$\begin{aligned} \partial_{tt}\mathbf{u} - \text{div}\,\mathbf{T}_s(\mathbf{u}) &= \mathbf{f}_s \quad \text{in} \quad \Omega_s(0) \\ \mathbf{u} &= 0 \quad \text{on} \quad \Gamma_s \\ \mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in} \quad \Omega_s(0) \\ \mathbf{T}_s(\mathbf{u}) &= \lambda\,\text{trace}\,\sigma(\mathbf{u})\mathbf{I} + 2\mu\sigma(\mathbf{u}), \quad \sigma(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^t). \end{aligned}$$

Some mathematical preliminaries: energy estimates

If $(\mathbf{v}, p)$ and $\mathbf{u}$ exist over the time interval $(0, T)$, then:

- multiply the Navier-Stokes equations for the fluid by $\mathbf{v}$ and integrate over $\Omega_f(t)$ to get

$$\int_{\Omega_f(t)} \partial_t \mathbf{v} \cdot \mathbf{v} + \int_{\Omega_f(t)} \mathbf{D}(\mathbf{v}) : \mathbf{D}(\mathbf{v}) - \int_{\Omega_f(t)} p \,\text{div}\, \mathbf{v} + \int_{\Omega_f(t)} (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot \mathbf{v}$$

$$= \int_{\Omega_f(t)} \mathbf{f}_f \mathbf{v} + \int_{\Gamma_{fs}(t)} \mathbf{T}_f(\mathbf{v}, p) \cdot \mathbf{n}_t \cdot \mathbf{v}.$$

- multiply the Navier-Lamé equations for the elastic structure by $\mathbf{u}$ and integrate over $\Omega_s(0) := \Omega_s$ to get

$$\int_{\Omega_s} \partial_{tt} \mathbf{u} \cdot \partial_t \mathbf{u} + (\lambda + 2\mu) \int_{\Omega_s} \sigma(\mathbf{u}) : \sigma(\partial_t \mathbf{u})$$

$$= \int_{\Omega_s} \mathbf{f}_s \partial_t \mathbf{u} + \int_{\Gamma_{fs}(0)} \mathbf{T}_s(\mathbf{u}) \cdot \mathbf{n} \cdot \partial_t \mathbf{u} + \int_{\Gamma_{fs}(t)} \mathbf{T}_s(\mathbf{u}) \cdot \mathbf{n}_t \cdot \partial_t \mathbf{u}.$$

Some mathematical preliminaries: energy estimates

- add the previous equations and use the interface conditions to obtain

$$\int_{\Omega_f(t)} \partial_t \mathbf{v}\cdot\mathbf{v} + 2\nu\int_{\Omega_f(t)} |\mathbf{D}(\mathbf{v})|^2 + \int_{\Omega_f(t)} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v} + \int_{\Omega_s} \partial_{tt}\mathbf{u}\cdot\partial_t\mathbf{u}$$

$$+(\lambda+2\mu)\int_{\Omega_s} \sigma(\mathbf{u}) : \sigma(\partial_t\mathbf{u}) = \int_{\Omega_f(t)} \mathbf{f}_f\mathbf{v} + \int_{\Omega_s} \mathbf{f}_s\partial_t\mathbf{u}.$$

- this can be rewritten as

$$\frac{1}{2}\int_{\Omega_f(t)} \partial_t|\mathbf{v}|^2 + \int_{\Omega_f(t)} |\mathbf{D}(\mathbf{v})|^2 + \int_{\Omega_f(t)} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}$$

$$+\frac{1}{2}\frac{d}{dt}\int_{\Omega_s} |\partial_t\mathbf{u}|^2 + \frac{1}{2}(\lambda+2\mu)\frac{d}{dt}\int_{\Omega_s} |\sigma(\mathbf{u})|^2 = \int_{\Omega_f(t)} \mathbf{f}_f\mathbf{v} + \int_{\Omega_s} \mathbf{f}_s\partial_t\mathbf{u}.$$

- now use Reynolds' transport theorem

$$\frac{d}{dt}\int_{\Omega_f(t)} |\mathbf{v}|^2 = \int_{\Omega_f(t)} \partial_t|\mathbf{v}|^2 + \int_{\Gamma_{fs}(t)} |\mathbf{v}|^2\cdot\mathbf{v}\cdot\mathbf{n}$$

Some mathematical preliminaries: energy estimates

and use the incompressibility condition to deduce

$$\int_{\Omega_f(t)} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}\int_{\Gamma_{fs}(t)} \mathbf{v}\cdot\mathbf{n}\cdot|\mathbf{v}|^2.$$

Summarizing, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_f(t)} |\mathbf{v}|^2 + 2\nu\int_{\Omega_f(t)} |\mathbf{D}(\mathbf{v})|^2 + \frac{1}{2}\frac{d}{dt}\int_{\Omega_s} |\partial_t \mathbf{u}|^2 + \frac{1}{2}(\lambda+2\mu)\frac{d}{dt}\int_{\Omega_s} |\sigma(\mathbf{u})|^2$$

$$= \int_{\Omega_f(t)} \mathbf{f}_f \mathbf{v} + \int_{\Omega_s} \mathbf{f}_s \partial_t \mathbf{u}.$$

Now integrate in time to deduce that

$$\|\mathbf{v}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_f(t)))} + \|\mathbf{v}\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f(t)))}$$

$$+\|\mathbf{u}\|^2_{\mathbf{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega_s))} + \|\mathbf{u}\|^2_{\mathbf{L}^\infty(0,T;\mathbf{H}^1(\Omega_s))} \le C,$$

where C is a constant depending on the data of the problem.

Some mathematical preliminaries: energy estimates

Remarks:

- The presence of the convective term in the Navier-Stokes equations is essential in the case of a time-moving interface. We have seen that his term has the role to balance the movement of the interface. Indeed, this is what we would expect, since in the Lagrangean form it enters in the expression of the total derivative and the latter commutes with the spatial integration over the reference fluid domain.

  Without this term there is no energy estimate, therefore the problem with the fluid equations lacking it is most probably not well posed, since we cannot recover the stability result obtained for the solution on the previous slide. This would also reflect itself in problems arising during discretization: there might not be a stable algorithm for the simulation of such a problem.

  $\Longrightarrow$ we cannot simplify the model involving a noncylindrical fluid domain upon just taking the Stokes equations instead of the Navier-Stokes ones!!!

  This has been first observed by Errate, Esteban & Maday in 1994 for a 1D fluid-structure interaction problem and confirmed ever since in many papers. We will encounter again this issue later on in these lectures.

Some mathematical preliminaries: energy estimates

Remarks (continued):

- The coupling conditions make sense both physically and mathematically. They have been set in a 'natural' way based on our common sense and during the process of deducing the a priori estimates it came out that they describe indeed (now from a mathematical point of view) the energy balance between the fluid and the structure.
- In order to derive the previous uniform bounds in the respective spaces we assumed that we were dealing with smooth enough domains. However, the regularity of the fluid-structure boundary is not granted: it is one of the unknowns of the problem. But here we were only deriving some a priori estimates... We shall see later how to handle the problem of an irregular domain.

A (short) survey of existence results

The stationary case:

Here the main difficulty relies on the different formulations of the fluid equations (Eulerian description) and of the elastic structure (Lagrangean description). An Euler-to-Lagrange transformation leads to a unified description, however the fluid equations become very complex, their coefficients depending on the structural displacement.

♣ Grandmont '98: study of a 2D/1D FSI problem, where a Stokes fluid is partially bounded by an elastic beam.

♣ Grandmont '02: study of a 3D/3D problem describing the interaction of a Navier-Stokes fluid with an elastic vessel. The latter completely encloses the fluid and is modelled by the nonlinear Navier-Lamé equations.

♣ Bayada, Chambat et al. '03: analysis of a 2D/1D Stokes-rod coupled problem with nonhomogeneous boundary conditions.

♣ S. '06: analysis of a 3D/3D Stokes flow in an elastic cylinder with thickness; the behavior of the structure is modelled by the linearized Navier-Lamé equations and periodic conditions are assumed at the ends of the cylinder.

♣ S. '07: study of a 3D/3D Navier-Stokes fluid in a flexible tube, whose walls are modelled by a (nonlinear) elastic structure. BCs: prescribed velocities for the fluid at the tube's ends, clamped outer boundary for the structure + conditions on the stresses at the tube's ends.

A (short) survey of existence results

The time dependent case:

- cylindrical domains
- time-moving domains

Cylindrical domains (infinitesimal displacements): there is a plethora of papers handling this situation. To make a selection, here we are only concerned with the case of a fluid through a flexible tube or contained in an elastic vessel.

- 2D fluid/1D structure:
  ♣ Canić & Mikelić '03: study of a creeping flow through a long tube with a membranar wall; the fluid is driven by a time-dependent pressure drop between the tube's ends; asymptotic techniques are used in order to obtain Biot-type equations for effective pressure and effective displacements.
- 3D fluid/2D structure:
  ♣ Flori & Orenga '98: a weakly viscous, compressible fluid interacting with a plate clamped on its entire boundary.
- 3D fluid/3D structure or higher dimensions:
  ♣ Lions '69: a transmission problem for a fluid in an elastic container.

A (short) survey of existence results

Cylindrical domains

- 3D fluid/3D structure or higher dimensions (continued):
  ♣ Du, Gunzburger et al. '03: mathematical analysis of a FSI problem where the fluid is completely contained in a vessel with flexible walls.
  ♣ S. '07: similar problem as above, but for a fluid through an elastic tube; nonstandard BCs involving the fluid pressure.

Time moving domains

- 1D fluid/1D structure:
  ♣ Errate, Esteban & Maday '94: analysis of a very simple FSI problem. In case of a noncylindrical domain the convective term in the fluid equations is essential.
- 2D fluid/1D structure:
  ♣ Flori & Orenga '99: study of a compressible, irrotational fluid interacting with an elastic plate which occupies a portion of the fluid's boundary.
  ♣ Beirao da Veiga '04: proof of a strong solution for the coupled FSI problem (Navier-Stokes equations coupled wit an elastic beam model). The technique therein allows to consider the Eulerian and the Lagrangean descriptions simultaneously; the change of variables is done all through the analysis (similarly to the numerical ALE approach).

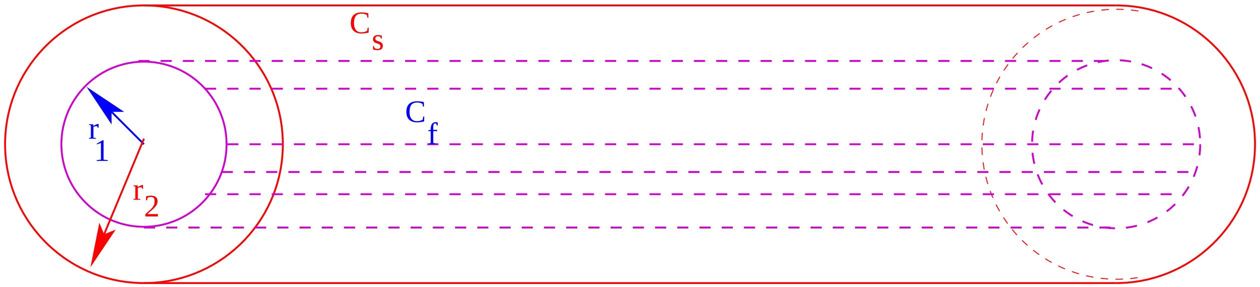
A (short) survey of existence results

Time moving domains (continued)

- 3D fluid/2D structure:
  ♣ Chambolle, Desjardins et al. '04: a FSI problem where the fluid is contained in a box with an elastic cover having the rest of the boundaries rigid and fixed: existence of a weak solution.
  ♣ S. '04: a similar problem as above, but with inflow and outflow sections for the fluid and with nonstandard boundary conditions involving the pressure.
  ♣ S. '04: study of a fluid inside a flexible cylindrical tube (Navier-Stokes equations for the fluid, cylindrical shell equations for the elastic structure): existence of a weak solution.

A result in the stationary case: problem setting

The steady-state flow of a Stokes fluid moving in an elastic tube with thickness: problem setting



$\tilde{C}_f := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ x_1^2 + x_2^2 < r_1^2\}$ is the infinite cylindrical pipe occupied by the viscous, incompressible fluid

$\tilde{C}_s := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ r_1^2 < x_1^2 + x_2^2 < r_2^2\}$ is the initial configuration of the elastic structure.

$\widetilde{Cyl} = \tilde{C}_f \cup \tilde{C}_s$

$\tilde{\Gamma}_{fs} := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ x_1^2 + x_2^2 = r_1^2\}$ is the fluid-structure interface.

$\tilde{\Gamma}_0$ is the exterior boundary of the elastic tube wall.

A result in the stationary case: problem setting

$$\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x}) = \mathbf{x} + \tilde{\mathbf{u}}(\mathbf{x})$$ (deformation of the fluid-structure interface).

$\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f)$ denotes the current (deformed) fluid domain.

Equations for the elastic structure:

$$\begin{aligned}
-\mathbf{div}(\lambda \text{trace } \mathbf{e}(\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u})) &= \mathbf{g} \text{ in } \tilde{C}_s \\
(\lambda \text{trace } \mathbf{e}(\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u})) \cdot \mathbf{n} &= \mathbf{G} \text{ on } \tilde{\Gamma}_{fs}, \\
\mathbf{u} &= 0 \text{ on } \tilde{\Gamma}_0 \\
\mathbf{u}(x_1, x_2, x_3) &= \mathbf{u}(x_1, x_2, x_3 + \frac{2\pi}{a}) \text{ in } \tilde{C}_s,
\end{aligned}$$

with $a \in \mathbf{R}_+^*$ constant, $a << 2\pi$. $\lambda > 0$, $\mu > 0$ are the Lamé constants and

$\mathbf{e}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t)$ is Green's strain tensor for the elastic material.

$\mathbf{G} := -\boldsymbol{\sigma}_\mathbf{f} \cdot \mathbf{n}$ is the surfacic force applied by the fluid on the interface

$\boldsymbol{\sigma}_\mathbf{f} := -p^\phi \cdot \mathbf{I} + 2\nu\mathbf{e}(\mathbf{v}^\phi)$ is the fluid stress tensor, and

$\mathbf{e}(\mathbf{v}^\phi) := \frac{1}{2}(\nabla\mathbf{v}^\phi + (\nabla\mathbf{v}^\phi)^t)$ is the fluid strain (to be written in the reference configuration, i.e. in the undeformed fluid domain).

A result in the stationary case: from Euler to Lagrange

Equations for the fluid flow:

$$\begin{aligned}
-\nu\Delta\mathbf{v}^{\tilde{\phi}} + \nabla p^{\tilde{\phi}} &= \mathbf{f}^{\tilde{\phi}} \text{ in } \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f) \\
\operatorname{div} \mathbf{v}^{\tilde{\phi}} &= 0 \text{ in } \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f) \\
\mathbf{v}^{\tilde{\phi}} &= 0 \text{ on } \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{\Gamma}_{fs}), \\
\mathbf{v}^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\mathbf{x})) &= \mathbf{v}^{\tilde{\phi}}(x_1, x_2, x_3 + \frac{2\pi}{a}), \quad \mathbf{x} = (x_1, x_2, x_3) \in \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f).
\end{aligned}$$

Problem: fluid equations set in Eulerian formulation, the elastic structure in Lagrangean description $\rightsquigarrow$ transformation, in order to get a unitary description!

We transform the unknown domain $\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\tilde{C}_f)$ into $\tilde{C}_f$ (which we know). We define

$$\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}) := Id + \mathcal{L}(\text{trace}_{fs}(\tilde{\mathbf{u}})),$$

where Id is the identity, trace_{fs} is the trace operator over $\tilde{\Gamma}_{fs}$ and $\mathcal{L} : \tilde{\Gamma}_{fs} \to \tilde{C}_f$ is a linear, continuous lifting. Denote $\mathbf{x}^{\tilde{\phi}} := \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x})$, $\mathbf{x} \in \tilde{C}_f$.

A result in the stationary case: from Euler to Lagrange

With the following transformations:

$$p^{\tilde{\phi}}(\mathbf{x}^{\tilde{\phi}}) = p^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x})) =: p(\mathbf{x}), \quad \mathbf{v}^{\tilde{\phi}}(\mathbf{x}^{\tilde{\phi}}) = \mathbf{v}^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x})) =: \mathbf{v}(\mathbf{x})$$

$$\mathbf{f}^{\tilde{\phi}}(\mathbf{x}^{\tilde{\phi}}) = \mathbf{f}^{\tilde{\phi}}(\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x})) =: \frac{\mathbf{f}(\tilde{\mathbf{u}}(\mathbf{x}))}{J(\tilde{\mathbf{u}})} \quad (J(\tilde{\mathbf{u}}) := \det\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}))$$

and $\mathbf{n}^{\tilde{\phi}} = \frac{\mathrm{cof}\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})\cdot\mathbf{n}}{||\mathrm{cof}\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})\cdot\mathbf{n}||}, \quad d\sigma^{\tilde{\phi}} = ||\mathrm{cof}\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})||d\sigma$

$$\mathbf{M} := \mathrm{cof}\nabla\tilde{\boldsymbol{\phi}} \text{ (cofactor matrix)}, \quad \mathbf{N} := (\nabla\tilde{\boldsymbol{\phi}})^{-1}\cdot\mathrm{cof}\nabla\tilde{\boldsymbol{\phi}},$$

the fluid system becomes (when written in the reference configuration):

$$\begin{aligned}
-\nu\,\mathrm{div}\,((\mathbf{N}\nabla)\mathbf{v}) + (\mathbf{M}\nabla)p &= \mathbf{f} \text{ in } \tilde{C}_f \\
\mathrm{div}\,(\mathbf{M}^t\mathbf{v}) &= 0 \text{ in } \tilde{C}_f \\
\mathbf{v} &= 0 \text{ on } \tilde{\Gamma}_{fs} \\
\mathbf{v}(\mathbf{x}) &= \mathbf{v}(x_1, x_2, x_3 + \frac{2\pi}{a}) \text{ in } \tilde{C}_f.
\end{aligned}$$

Keep in mind that the functions above are related to the (initial) displacement $\tilde{\mathbf{u}}$, however we omit it in the writing.

A result in the stationary case: from cylinder to torus

Let T be a torus in $\mathbf{R}^3$. We transform the cylinders

$C_f := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 0 < x_3 < \frac{2\pi}{a}, \ x_1^2 + x_2^2 < r_1^2\}$ and

$C_s := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 0 < x_3 < \frac{2\pi}{a}, \ r_1^2 < x_1^2 + x_2^2 < r_2^2\}$

(having the interface $\Gamma_{fs} := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \ : \ 0 < x_3 < \frac{2\pi}{a}, \ x_1^2 + x_2^2 = r_1^2\}$)

into the tori T_f, respectively T_s, upon identifying:

- the disk $\{x_3 = 0, \ x_1^2 + x_2^2 \leq r_1^2\}$ with the disk $\{x_3 = \frac{2\pi}{a}, \ x_1^2 + x_2^2 \leq r_1^2\}$
- $\{x_3 = 0, \ r_1^2 \leq x_1^2 + x_2^2 \leq r_2^2\}$ with $\{x_3 = \frac{2\pi}{a}, \ r_1^2 \leq x_1^2 + x_2^2 \leq r_2^2\}$.

We consider the mapping $[0, L] \ni s \mapsto \delta(s) = \varphi, \ \delta(s) = \frac{2\pi s}{L}, \ \forall s \in [0, L]$, where we take $L = \frac{2\pi}{a}$.

Then the mapping transforming the cylinder $Cyl = C_f \cup C_s$ into the torus is of the form:

$$\mathbf{t} : Cyl \subset \mathbf{R}^3 \to T \subset \mathbf{R}^3$$

$$t_1(x) = (\frac{1}{a} + x_1)\cos(ax_3); \ t_2(x) = (\frac{1}{a} + x_1)\sin(ax_3), \ t_3(x) = x_2.$$

A result in the stationary case: from cylinder to torus

Then the fluid system is equivalent to

$$\begin{aligned} -\nu \text{div}\,((\mathcal{N}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f \cdot \nabla)\mathbf{v}(\tilde{\mathbf{u}}) \cdot \boldsymbol{\gamma}^f) &+ \nu(\mathcal{N}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f \cdot \nabla)\mathbf{v}(\tilde{\mathbf{u}}) \cdot \text{div}\,\boldsymbol{\gamma}^f \\ &+ (\mathcal{M}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f \cdot \nabla)p(\tilde{\mathbf{u}}) = \widetilde{\mathbf{f}^J}(\tilde{\mathbf{u}}) \text{ in } T_f \\ \mathcal{M}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f : \nabla \mathbf{v}(\tilde{\mathbf{u}}) &= 0 \text{ in } T_f \\ \mathbf{v}(\tilde{\mathbf{u}}) &= 0 \text{ on } \partial T_f, \end{aligned}$$

where $\mathcal{M}(\tilde{\mathbf{u}})(\mathbf{X}) := \text{cof}\,(\boldsymbol{\gamma}^f \cdot \nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}(\mathbf{X})))$,

$\mathcal{N}(\tilde{\mathbf{u}})(\mathbf{X}) := (\boldsymbol{\gamma}^f \cdot \nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}))^{-1} \cdot \text{cof}\,(\boldsymbol{\gamma}^f \cdot \nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}))$, and

$\gamma^f_{ij}(\mathbf{x}) := \frac{\partial t_j}{\partial x_i}(\mathbf{x}), i, j = 1, 2, 3$ (we make the notation $\mathbf{t}(\mathbf{x}) = \mathbf{X}$ and $\gamma^f_{ij}(\mathbf{X}) := \gamma^f_{ij} \circ \mathbf{t}^{-1}(\mathbf{X})$).

We also have

$$\begin{aligned} \widetilde{\mathbf{f}^J}(\tilde{\mathbf{u}})(\mathbf{X}) &:= (\mathbf{f}(\tilde{\mathbf{u}}) \circ \mathbf{t}^{-1})(\mathbf{X})J((\boldsymbol{\gamma}^f\nabla)\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}(\mathbf{X}))); \\ \mathbf{v}(\tilde{\mathbf{u}})(\mathbf{X}) &:= (\mathbf{v}(\tilde{\mathbf{u}}) \circ \mathbf{t}^{-1})(\mathbf{X}); \\ p(\tilde{\mathbf{u}})(\mathbf{X}) &:= (p(\tilde{\mathbf{u}}) \circ \mathbf{t}^{-1})(\mathbf{X}). \end{aligned}$$

A result in the stationary case: from cylinder to torus

Analogously, the elastic structure system is equivalent to:

$$\begin{aligned}
-\text{div}\,(\lambda \text{tr}\,\mathbf{E}(\boldsymbol{\gamma}^s \cdot (\nabla \mathbf{U}(\tilde{\mathbf{u}}))^t) \cdot \mathbf{I} + 2\mu \mathbf{E}(\boldsymbol{\gamma}^s \cdot (\nabla \mathbf{U}(\tilde{\mathbf{u}}))^t)) &= K(\boldsymbol{\gamma}^s)^{-1} \cdot \mathbf{g} \text{ in } T_s \\
(\lambda \text{tr}\,\mathbf{E}(\boldsymbol{\gamma}^s \cdot (\nabla \mathbf{U}(\tilde{\mathbf{u}}))^t) \cdot \mathbf{I} + 2\mu \mathbf{E}(\boldsymbol{\gamma}^s \cdot (\nabla \mathbf{U}(\tilde{\mathbf{u}}))^t) \cdot \mathbf{n} &= K\mathbf{G} \text{ on } \Gamma_{fs} \\
\mathbf{U}(\tilde{\mathbf{u}}) &= 0 \text{ on } \Gamma_0,
\end{aligned}$$

where $K := \lambda + \mu$ and $\mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla \mathbf{U}^t) := \frac{1}{2}(\boldsymbol{\gamma}^s \cdot \nabla \mathbf{U}^t + (\boldsymbol{\gamma}^s \cdot \nabla \mathbf{U}^t)^t)$.

Γ_0 denotes the exterior boundary surface of the elastic torus.

Observe that $\mathbf{G}(\tilde{\mathbf{u}}) = p(\tilde{\mathbf{u}})\mathcal{M}(\tilde{\mathbf{u}}) \cdot \mathbf{n} - \nu(\mathcal{N}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f \nabla)\mathbf{v}(\tilde{\mathbf{u}}) \cdot \mathbf{n}$.

Here we make the same convention of notation as on the previous slide, where

$$\gamma^s_{mj}(\mathbf{x}) = (\lambda + \mu)\frac{\partial t_j}{\partial x_m}(\mathbf{x}), \quad j, m = 1, 2, 3$$

and

$$\mathbf{g}(\mathbf{X}) = (\mathbf{g} \circ \mathbf{t}^{-1})(\mathbf{X}), \;\; \mathbf{G}(\mathbf{X}) = (\mathbf{G} \circ \mathbf{t}^{-1})(\mathbf{X}).$$

A result in the stationary case: the fluid problem

Let $p \in \mathbf{R}$ with $3 < p < \infty$. Consider the following system:

$$
\begin{aligned}
-\nu \text{div}\,((\mathcal{N}\boldsymbol{\gamma}^f \cdot \nabla)\mathbf{v} \cdot \boldsymbol{\gamma}^f) \quad &+ \quad \nu(\mathcal{N}\boldsymbol{\gamma}^f \cdot \nabla)\mathbf{v} \cdot \text{div}\,\boldsymbol{\gamma}^f \\
&+ \quad (\mathcal{M}\boldsymbol{\gamma}^f \cdot \nabla)p = \hat{\mathbf{f}} \quad \text{in } T_f \\
\mathcal{M}\boldsymbol{\gamma}^f : \nabla\mathbf{v} \quad &= \quad 0 \quad \text{in } T_f \\
\mathbf{v} \quad &= \quad 0 \quad \text{on } \partial T_f,
\end{aligned}
$$

along with the hypotheses:

(H_1) $\mathcal{N}$ is a symmetric and positive definite matrix such that coeff $(\mathcal{N}) \in \mathbf{W}^{1,p}(T_f)$, $\boldsymbol{\gamma}^f$ is a regular enough matrix; also assume that $\exists c > 0$ a constant such that $\mathcal{N}\boldsymbol{\gamma}^f \geq c\mathbf{I}$;

(H_2) $\mathcal{M}$ is invertible in $\mathbf{W}^{1,p}(T_f)$ and $\exists\ \boldsymbol{\Theta}$ with $\mathcal{M} =$ cof $\nabla\boldsymbol{\Theta}$

(H_3) $\exists\ C > 0$ a constant with $||\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f||_{\mathbf{W}^{1,p}(T_f)} \leq C$, $||\mathbf{I} - (\mathcal{M}\boldsymbol{\gamma}^f)^t||_{\mathbf{W}^{1,p}(T_f)} \leq C$ and $||\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f||_{\mathbf{W}^{1,p}(T_f)} \leq C$.

Theorem F. Let $\hat{\mathbf{f}} \in \mathbf{L}^p(T_f)$. There exists a unique solution $(\mathbf{v}, p)$ of the above system in $(\mathbf{W}^{2,p}(T_f) \cap \mathbf{W}^{1,p}_{0,\partial T_f}(T_f)) \times W^{1,p}(T_f)$, with:

$$||\mathbf{v}||_{\mathbf{W}^{2,p}(T_f)} + ||p||_{W^{1,p}(T_f)} \leq C_1 ||\hat{\mathbf{f}}||_{\mathbf{L}^p(T_f)}.$$

A result in the stationary case: the fluid problem

Sketch of the proof:

- The existence of a unique solution $(\mathbf{v}, p) \in \mathbf{H}^1_{0,\Gamma_{fs}}(T_f) \times L^2_0(T_f)$ to the fluid system above follows in the usual way. For the existence of a unique pressure verify an adequate *inf-sup condition*:

$$\exists k > 0 \text{ (constant) s.t.} \sup_{\boldsymbol{\psi} \in \mathbf{H}^1_0(T_f)} \frac{\int_{T_f} \tau \mathcal{M} \boldsymbol{\gamma}^f : \nabla \boldsymbol{\psi}}{||\boldsymbol{\psi}||_{\mathbf{H}^1_0(T_f)}} \geq k ||\tau||_{L^2(T_f)}, \ \forall \tau \in L^2_0(T_f).$$

Indeed, $\forall\, \tau \in L^2_0(T_f)\ \exists\, \widehat{\boldsymbol{\psi}} \in \mathbf{H}^1_0(T_f)$ s.t. $\text{div}\, \widehat{\boldsymbol{\psi}} = \tau$ and $||\widehat{\boldsymbol{\psi}}||_{\mathbf{H}^1(T_f)} \leq C ||\tau||_{L^2(T_f)}$.

Associate to any given $\tau \in L^2_0(T_f)$ a $\widehat{\boldsymbol{\psi}}$ and take $\boldsymbol{\psi}$ s.t. $\nabla \boldsymbol{\psi} = (\mathcal{M}\boldsymbol{\gamma}^f)^{-t} \nabla \widehat{\boldsymbol{\psi}}$.

Then $\boldsymbol{\psi} \in \mathbf{H}^1_0(T_f)$ and we can use the above estimate for $||\widehat{\boldsymbol{\psi}}||$ to deduce the inf-sup condition with the constant k depending on $||(\mathcal{M}\boldsymbol{\gamma}^f)^{-t}||_{\mathbf{L}^\infty(T_f)}$.

- We prove the stated regularity upon constructing a Cauchy sequence of solutions to the fluid system above and showing that it converges to the unique solution.
- The estimations obtained in the previous step allow to verify the inequality stated in the Theorem.

A result in the stationary case: the fluid problem

Consider the sequence $S(n)$:

$$
\begin{aligned}
-\nu \mathrm{div}(\nabla \mathbf{v}^n \cdot \boldsymbol{\gamma}^f) &+ \nu \nabla \mathbf{v}^n \cdot \mathrm{div}\boldsymbol{\gamma}^f + \nabla p^n = \hat{\mathbf{f}} - \nu \mathrm{div}(((\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f)\nabla)\mathbf{v}^{n-1}) \cdot \boldsymbol{\gamma}^f) \\
&+ \nu((\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f)\nabla)\mathbf{v}^{n-1} \cdot \mathrm{div}\, \boldsymbol{\gamma}^f + (\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f)\nabla p^{n-1} \quad \text{in } T_f \\
\mathbf{I} : \nabla \mathbf{v}^n &= (\mathbf{I} - \mathbf{M}\boldsymbol{\gamma}^f) : \nabla \mathbf{v}^{n-1} \quad \text{in } T_f \\
\mathbf{v}^n &= 0 \quad \text{on } \Gamma_{fs},
\end{aligned}
$$

with the first term $S(0)$:

$$
\begin{aligned}
-\nu \mathrm{div}(\nabla \mathbf{v}^0 \cdot \boldsymbol{\gamma}^f) + \nu \nabla \mathbf{v}^0 \cdot \mathrm{div}\, \boldsymbol{\gamma}^f + \nabla p^0 &= \hat{\mathbf{f}} \quad \text{in } T_f \\
\mathrm{div}\, \nabla \mathbf{v}^0 &= 0 \quad \text{in } T_f \\
\mathbf{v}^0 &= 0 \quad \text{on } \Gamma_{fs}.
\end{aligned}
$$

$\exists!$ solution $\mathbf{v}^0 \in \mathbf{W}^{2,p}(T_f),\ p^0 \in W^{1,p}(T_f)$.

Moreover, $\forall\, n \in \mathbb{N}$, $(\mathbf{v}^n, p^n) \in (\mathbf{W}^{2,p}(T_f) \cap \mathbf{W}_0^{1,p}(T_f)) \times W^{1,p}(T_f)$, by mathematical induction on n.

A result in the stationary case: the fluid problem

(v^n, p^n) is a Cauchy sequence in $\mathbf{W}^{2,p}(T_f) \times W^{1,p}(T_f)$:

Calculate $S(n+1) - S(n)$:

$-\nu \text{div}(\nabla(\mathbf{v}^{n+1} - \mathbf{v}^n) \cdot \boldsymbol{\gamma}^f) + \nu \nabla(\mathbf{v}^{n+1} - \mathbf{v}^n) \text{div}\, \boldsymbol{\gamma}^f + \nabla(p^{n+1} - p^n)$

$= -\nu \text{div}(((\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f)\nabla)(\mathbf{v}^n - \mathbf{v}^{n-1}) \cdot \boldsymbol{\gamma}^f) + \nu((\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f)\nabla)(\mathbf{v}^n - \mathbf{v}^{n-1}) \text{div}\, \boldsymbol{\gamma}^f$

$$
\begin{aligned}
& + && ((\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f)\nabla)(p^n - p^{n-1}) \quad \text{in } T_f \\
\text{div}\,(\mathbf{v}^{n+1} - \mathbf{v}^n) & = && (\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f) : \nabla(\mathbf{v}^n - \mathbf{v}^{n-1}) \quad \text{in } T_f \\
\mathbf{v}^{n+1} - \mathbf{v}^n & = && 0 \quad \text{on } \Gamma_{fs}.
\end{aligned}
$$

For this Stokes problem we get the estimates

$$
\begin{aligned}
||\mathbf{v}^{n+1} - \mathbf{v}^n||_{\mathbf{W}^{2,p}(T_f)} & + && ||p^{n+1} - p^n||_{W^{1,p}(T_f)} \\
& \leq && \text{const}\,\{||\mathbf{I} - \mathcal{N}\boldsymbol{\gamma}^f||_{\mathbf{W}^{1,p}(T_f)} ||\mathbf{v}^n - \mathbf{v}^{n-1}||_{\mathbf{W}^{2,p}(T_f)} \\
& + && ||\mathbf{I} - (\mathcal{M}\boldsymbol{\gamma}^f)^t||_{\mathbf{W}^{1,p}(T_f)} ||\mathbf{v}^n - \mathbf{v}^{n-1}||_{\mathbf{W}^{2,p}(T_f)} \\
& + && ||\mathbf{I} - \mathcal{M}\boldsymbol{\gamma}^f||_{\mathbf{W}^{1,p}(T_f)} ||p^n - p^{n-1}||_{W^{1,p}(T_f)}\},
\end{aligned}
$$

const > 0 being a constant independent on $n, \mathcal{N},\ \mathcal{M}$, but depending on the bound of $\boldsymbol{\gamma}^f$.

A result in the stationary case: the fluid problem

Now choose C in the hypotheses we made in order to satisfy const $\cdot C < 1$; it follows that:

$$||\mathbf{v}^{n+1} - \mathbf{v}^n||_{\mathbf{W}^{2,p}(T_f)} + ||p^{n+1} - p^n||_{W^{1,p}(T_f)}$$

$$\leq C_{prod}\{||\mathbf{v}^n - \mathbf{v}^{n-1}||_{\mathbf{W}^{2,p}(T_f)} + ||p^n - p^{n-1}||_{W^{1,p}(T_f)}\},$$

with $0 < C_{prod} < 1$.

Consequently, the sequence $(\mathbf{v}^n, p^n)$ converges in $\mathbf{W}^{2,p}(T_f) \times W^{1,p}(T_f)$. Thus, there exists the limit $(\mathbf{v}_C, p_C) \in (\mathbf{W}^{2,p}(T_f) \cap \mathbf{W}_0^{1,p}(T_f)) \times W^{1,p}(T_f)$ such that

$$\mathbf{v}^n \to \mathbf{v}_C \text{ in } \mathbf{W}^{2,p}(T_f) \text{ as } n \to \infty$$

and

$$p^n \to p_C \text{ in } W^{1,p}(T_f) \text{ as } n \to \infty.$$

By passing now to the limit in $S(n)$, we conclude that $(\mathbf{v}_C, p_C)$ is the unique solution of our system and thus$\mathbf{v}_C = \mathbf{v}$ and $p_C = p$.

With the above estimations we can write

$$||\mathbf{v}^n||_{\mathbf{W}^{2,p}(T_f)} + ||p^n||_{W^{1,p}(T_f)} \leq C||\hat{\mathbf{f}}||_{\mathbf{L}^p(T_f)} + C_{prod}\{||\mathbf{v}^{n-1}||_{\mathbf{W}^{2,p}(T_f)} + ||p^{n-1}||_{W^{1,p}(T_f)}\},$$

thus for $n \to \infty$ we get $||\mathbf{v}||_{\mathbf{W}^{2,p}(T_f)} + ||p||_{W^{1,p}(T_f)} \leq C(1 - C_{prod})^{-1}||\hat{\mathbf{f}}||_{\mathbf{L}^p(T_f)}$.

A result in the stationary case: the structure problem

Consider the following system for the flexible structure:

$$
\begin{aligned}
-\text{div}\,(\lambda \text{tr}\,\mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla U^t)\cdot \mathbf{I} + 2\mu \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla U^t)) &= K(\boldsymbol{\gamma}^s)^{-1}\mathbf{g} \quad \text{in } T_s \\
(\lambda \text{tr}\,\mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla U^t)\cdot \mathbf{I} + 2\mu \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla U^t))\cdot \mathbf{n} &= K\mathbf{G} \quad \text{on } \Gamma_{fs} \\
\mathbf{U} &= 0 \quad \text{on } \Gamma_0,
\end{aligned}
$$

where $\mathbf{g}$ is a given volumic force and $\mathbf{G}$ is a given surfacic force (related to the fluid stress tensor).

Theorem S. For $p \in \mathbf{R},\ 3 < p < \infty$ let $\mathbf{g} \in L^p(T_s)$ and $\mathbf{G} \in W^{1-1/p,p}(\Gamma_{fs})$. Then there exists a unique solution $\mathbf{U} \in W^{2,p}(T_s) \cap W^{1,p}_{0,\Gamma_0}(T_s)$ of the above system and it satisfies:

$$||\mathbf{U}||_{W^{2,p}(T_s)} \leq \text{const}\ (||\mathbf{g}||_{L^p(T_s)} + ||\mathbf{G}||_{W^{1-1/p,p}(\Gamma_{fs})}).$$

A result in the stationary case: the structure problem

For the proof we can write the system in the form

$$\mathbf{A}(\mathbf{U}, \boldsymbol{\psi}) = \mathbf{L}(\boldsymbol{\psi}), \ \forall \boldsymbol{\psi} \in \mathbf{V},$$

where

$$\mathbf{A}(\mathbf{U}, \boldsymbol{\psi}) := \int_{T_s} \mathbf{S}(\mathbf{U}) : \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla \boldsymbol{\psi}^t),$$

with

$$\mathbf{S}(\mathbf{U}) := \lambda \mathrm{tr}\, \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla \mathbf{U}^t)\mathbf{I} + 2\mu \mathbf{E}(\boldsymbol{\gamma}^s \cdot \nabla \mathbf{U}^t)$$

and

$$\mathbf{L}(\boldsymbol{\psi}) := \int_{T_s} K\mathbf{g} \cdot \boldsymbol{\psi} d\mathbf{y} + \int_{\Gamma_{fs}} K\mathbf{G} \cdot \boldsymbol{\psi} d\sigma.$$

Denote $\mathbf{V} := \{\boldsymbol{\psi} \in H^1(T_s) \ : \ \boldsymbol{\psi} = 0 \text{ on } \Gamma_0\} = \mathbf{H}^1_{0,\Gamma_0}(T_s)$.

$\mathbf{A}$ is a continuous, bilinear form that is also $\mathbf{V}$-elliptic (via Korn's inequality) and $\mathbf{L}$ is a continuous linear form defined on $\mathbf{V}$. The existence of a unique solution follows by the Lax-Milgam lemma. For $\mathbf{U} \in W^{2,p}(T_s)$ and the estimate in the Theorem apply Th. 6.3.6 in Ciarlet '86 and the remarks after it.

A result in the stationary case: the coupled problem

Theorem. Let $p \in \mathbf{R}$ with $3 < p < \infty$, $\mathbf{f}^{\tilde{\phi}} \in L^p(\mathbf{R}^3)$ and $\mathbf{g} \in L^p(T_s)$. Assume there exists a constant $\chi > 0$ with:

$$C_{coupl}(\|\mathbf{f}^{\tilde{\phi}}\|_{L^p(\mathbf{R}^3)} + \|\mathbf{g}\|_{L^p(T_s)}) \leq \chi.$$

Then there exists a solution $(\mathbf{v}, p, \mathbf{U})$ of the coupled problem on the torus $T = T_f \cup T_s$, with $\mathbf{v} \in W^{2,p}(T_f) \cap W_0^{1,p}(T_f)$, $p \in W^{1,p}(T_f)$ and $\mathbf{U}$ sufficiently small in $W^{2,p}(T_s)$.

Idea of the proof: let

$$\mathcal{U}_\chi := \{\tilde{\mathbf{u}} \in W^{2,p}(T_s) \ : \ \|\tilde{\mathbf{u}}\|_{W^{2,p}(T_s)} \leq \chi\}.$$

The mapping

$$\mathcal{U}_\chi \ni \tilde{\mathbf{u}} \overset{\mathcal{A}}{\mapsto} \mathbf{U}(\tilde{\mathbf{u}}) \in W^{2,p}(T_s)$$

has at least one fixed point.

A result in the stationary case: the coupled problem

Let $\tilde{\mathbf{u}} \in \mathcal{U}_\chi$. Then $\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})$ is an invertible matrix in $\mathbf{W}^{1,p}(T_f)$ ($p > 3$) and (for χ sufficiently small) we have $\det\nabla\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}})(\mathbf{x}) > 0$, thus the deformation $\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}) = Id + \tilde{\mathbf{u}}$ is orientation preserving and injective.

Indeed, by the mean value theorem:

$$\begin{aligned} ||\tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}(\mathbf{x}_1) - \tilde{\boldsymbol{\phi}}(\tilde{\mathbf{u}}(\mathbf{x}_2)|| &= ||\mathbf{x}_1 - \mathbf{x}_2 + \tilde{\mathbf{u}}(\mathbf{x}_1) - \tilde{\mathbf{u}}(\mathbf{x}_2)|| \\ &\geq ||\mathbf{x}_1 - \mathbf{x}_2|| - \sup ||\nabla\tilde{\mathbf{u}}|| \cdot ||\mathbf{x}_1 - \mathbf{x}_2|| \\ &> (1 - C(T_f))||\mathbf{x}_1 - \mathbf{x}_2|| \text{ (for } \mathbf{x}_1 \neq \mathbf{x}_2), \end{aligned}$$

$C(T_f)$ being the constant in the orientation preserving theorem.

The solution $(\mathbf{v}(\tilde{\mathbf{u}}), p(\tilde{\mathbf{u}}))$ of the actual fluid equations satisfies the same type of equations as those in Theorem F for the similar Stokes system, with $\hat{\mathbf{f}} := \widetilde{\mathbf{f}^J}$, $\mathcal{N} := \mathcal{N}(\tilde{\mathbf{u}})$, $\mathcal{M} := \mathcal{M}(\tilde{\mathbf{u}})$.

Due to the smoothness of $\boldsymbol{\gamma}^f$, the hypotheses of Theorem F are satisfied for $\mathcal{N}(\tilde{\mathbf{u}})$ and $\mathcal{M}(\tilde{\mathbf{u}})$, too. Thus $\forall\ \tilde{\mathbf{u}} \in \mathcal{U}_\chi$, $(\mathbf{v}(\tilde{\mathbf{u}}), p(\tilde{\mathbf{u}})) \in W^{2,p}(T_f) \times W^{1,p}(T_f)$ and

$$\|\mathbf{v}(\tilde{\mathbf{u}})\|_{W^{2,p}(T_f)} + \|p(\tilde{\mathbf{u}})\|_{W^{1,p}(T_f)} \leq C_1 \|\widetilde{\mathbf{f}^J}(\tilde{\mathbf{u}})\|_{L^p(T_f)},$$

thus also

$$\|\mathbf{v}(\tilde{\mathbf{u}})\|_{W^{2,p}(T_f)} + \|p(\tilde{\mathbf{u}})\|_{W^{1,p}(T_f)} \leq C(C_1, \chi)\|\mathbf{f}^{\tilde{\phi}}\|_{L^p(\mathbf{R}^3)}.$$

A result in the stationary case: the coupled problem

Now $\mathbf{G}(\tilde{\mathbf{u}}) = p(\tilde{\mathbf{u}})\mathcal{M}(\tilde{\mathbf{u}}) \cdot \mathbf{n} - \nu(\mathcal{N}(\tilde{\mathbf{u}})\boldsymbol{\gamma}^f \nabla)\mathbf{v}(\tilde{\mathbf{u}}) \cdot \mathbf{n} \in W^{1-1/p,p}(\Gamma_{fs})$.

$\mathbf{G}$ in the right hand side of the structure equations for $\mathbf{U}$ (previous slides) satisfies:

$$\begin{aligned} \|\mathbf{G}\|_{W^{1-1/p,p}(\Gamma_{fs})} &\leq C(\|\mathbf{v}(\tilde{\mathbf{u}})\|_{W^{2,p}(T_f)} + \|p(\tilde{\mathbf{u}})\|_{W^{1,p}(T_f)}) \\ &\leq C(C_1,\chi)\|\mathbf{f}^{\tilde{\phi}}\|_{L^p}. \end{aligned}$$

Now apply Theorem S to get the existence of a unique solution $\mathbf{U}(\tilde{\mathbf{u}}) \in W^{2,p}(T_s)$ to the actual structure equations with

$$\|\mathbf{U}(\tilde{\mathbf{u}})\|_{W^{2,p}(T_s)} \leq const(C(C_1,\chi)\|\mathbf{f}^{\tilde{\phi}}\|_{L^p} + \|\mathbf{g}\|_{L^p(T_s)}).$$

We have thus constructed the mapping $\mathcal{U}_\chi \ni \tilde{\mathbf{u}} \overset{\mathcal{A}}{\mapsto} \mathbf{U}(\tilde{\mathbf{u}}) \in \mathcal{U}_\chi \subset W^{2,p}(T_s)$.

A result in the stationary case: the coupled problem

The mapping $\mathcal{A}$ has a fixed point, by the theorem of Schauder:

- $\mathcal{A}$ is weakly sequentially continuous on $W^{2,p}(T_s)$.
- $\mathcal{A}(\mathcal{U}_\chi) \subset \mathcal{U}_\chi$.
- $\mathcal{U}_\chi$ is convex and weakly compact in $W^{2,p}(T_s)$.

Indeed, let $\tilde{\mathbf{u}}_n \in \mathcal{U}_\chi$ with $\tilde{\mathbf{u}}_n \overset{n\to\infty}{\rightharpoonup} \tilde{\mathbf{u}}$ in $W^{2,p}(T_s)$.

By the previous estimates, $(\mathbf{v}(\tilde{\mathbf{u}}_n), p(\tilde{\mathbf{u}}_n), \mathbf{U}(\tilde{\mathbf{u}}_n))$ is (independently on n) bounded in $W^{2,p}(T_f) \times W^{1,p}(T_f) \times W^{2,p}(T_s)$, thus $\exists (\hat{\mathbf{v}}, \hat{p}, \hat{\mathbf{U}}) \in W^{2,p}(T_f) \times W^{1,p}(T_f) \times W^{2,p}(T_s)$ and $\exists\ (\tilde{\mathbf{u}}_{nk})_k \subset (\tilde{\mathbf{u}}_n)_n$ with

$$\mathbf{v}(\tilde{\mathbf{u}}_{nk}) \overset{k\to\infty}{\rightharpoonup} \hat{\mathbf{v}} \text{ in } W^{2,p}(T_f)$$

$$p(\tilde{\mathbf{u}}_{nk}) \overset{k\to\infty}{\rightharpoonup} \hat{p} \text{ in } W^{1,p}(T_f)$$

$$\mathbf{U}(\tilde{\mathbf{u}}_{nk}) \overset{k\to\infty}{\rightharpoonup} \hat{\mathbf{U}} \text{ in } W^{2,p}(T_s).$$

Show that $\mathbf{U}(\tilde{\mathbf{u}}) = \hat{\mathbf{U}}$ and this will prove the weak continuity of $\mathcal{A}$, since then $\mathbf{U}(\tilde{\mathbf{u}}_n) \rightharpoonup \mathbf{U}(\tilde{\mathbf{u}})$ in the weak topology of $W^{2,p}(T_s)$.

A result in the stationary case: the coupled problem

Remember that the exstence of a fixed point for $\mathcal{A}$ ensures the existence of a solution to the coupled problem on the torus.

Now transforming back to the original domain, we get the following result for the fluid-structure interaction problem in the cylinder of length $L = \frac{2\pi}{a}$:

Theorem. Let $\mathbf{f}^{\tilde{\phi}} \in \mathbf{L}^p(\mathbf{R}^3)$ and $\mathbf{g} \in \mathbf{L}^p_{per}(C_s)$. Assume there exists a constant $\chi_1 > 0$ with:

$$K(\|\mathbf{f}^{\tilde{\phi}}\|_{\mathbf{L}^p(\mathbf{R}^3)} + \|\mathbf{g}\|_{\mathbf{L}^p(C_s)}) \leq \chi_1,$$

where K is a constant depending on a.

Then there exists a solution $(\mathbf{v}, p, \mathbf{U})$ of the coupled FSI problem, with $\mathbf{v} \in \mathbf{W}^{2,p}_{per}(C_f) \cap \mathbf{W}$, $p \in W^{1,p}_{per}(C_f)$ and $\mathbf{U}$ sufficiently small in $\mathbf{W}^{2,p}_{per}(C_s)$.

A result in the stationary case: some function spaces

Let $\tilde{C}$ an infinite cylindrical pipe with boundary $\tilde{\Gamma}$. For $a \in \mathbf{R}_+ - \{0\}$ and the finite cylinder C with boundary Γ, define

$$\begin{aligned}
C^\infty_{0,per}(C) &:= \{f \in C^\infty_{per}(C) \ : \ \text{supp}\,(f) \cap (\bar{C} - \Gamma) \text{ is compact in } C_f\},\\
L^p_{per}(C) &:= \text{the closure of } C^\infty_{per}(C) \text{ in } L^p(C)\\
W^{m,p}_{per}(C) &:= \text{the closure of } C^\infty_{per}(C) \text{ in } W^{m,p}(C),\\
W^{m,p}_{0,per}(C) &:= \text{the closure of } C^\infty_{0,per}(C) \text{ in } W^{m,p}(C),\\
\tilde{\mathbf{W}} &:= \{\mathbf{F} \in C^\infty_{0,per}(C) : \nabla \cdot \mathbf{F} = 0\},\\
\mathbf{W} &:= \text{the closure of } \tilde{\mathbf{W}} \text{ in } W^{1,p}_{0,per}(C).
\end{aligned}$$

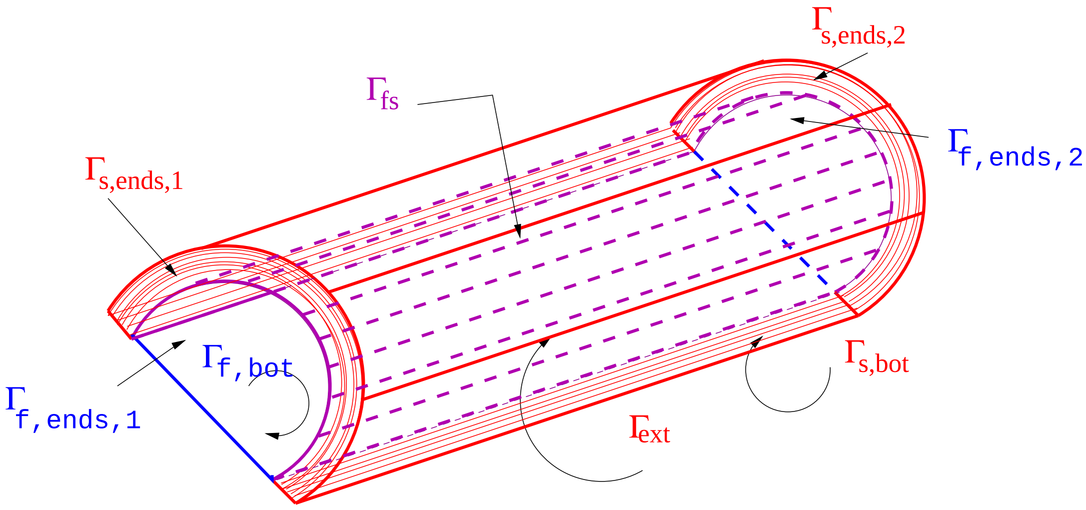
Observe that

$$\mathbf{W} = \{\mathbf{v} \in W^{1,p}_{0,per}(C) \ : \ \nabla \cdot \mathbf{v} = 0\}.$$

By the Poincaré inequality, the inner product in $W^{1,p}_{0,per}(C)$ is equivalent to the inner product

$$((\mathbf{v}, \mathbf{w})) := \int_C \frac{\partial v_j}{\partial x_i} \frac{\partial w_j}{\partial x_i} d\mathbf{x}, \ \ \mathbf{v}, \mathbf{w} \in W^{1,p}_{0,per}(C).$$

A result in the time-dependent case (cylind. domain)



Total domain: $\Omega = \Omega_f \cup \Omega_s$. $\Omega_f := S_f \times (0, L)$, $\Omega_s := S_s \times (0, L)$, where

$$S_f =: \{(a\cos\theta, a\sin\theta) : a \in (0, r),\ \theta \in (0, \pi)\}$$

$$S_s =: \{(a\cos\theta, a\sin\theta) : a \in (r, R),\ \theta \in (0, \pi)\} \qquad r < R,$$

Boundaries:

- Γ_{fs}: the fluid-structure interface,
- $\Gamma_{f,ends,k}$ $(k = 1, 2)$: the fluid boundaries at the ends of the tube,
- Γ_{ext}: the exterior lateral boundary of the elastic cylinder,
- $\Gamma_{s,ends} = \Gamma_{s,ends,1} \cup \Gamma_{s,ends,2}$: boundaries of the elastic cylinder at the tube's ends,
- $\Gamma_{bot} = \Gamma_{f,bot} \cup \Gamma_{s,bot}$ be the bottom part of the boundary.

Stokes flow through an elastic tube: the equations

The elastic structure:

$$\begin{aligned}\partial_{tt}\mathbf{u} - \lambda\nabla(\text{div }\mathbf{u}) - 2\mu\nabla\cdot\mathbf{e}(\mathbf{u}) &= \mathbf{g} \text{ in } (0,T)\times\Omega_s\\ \mathbf{u} &= 0 \text{ on } (0,T)\times(\Gamma_{ext}\cup\Gamma_{s,ends}\cup\Gamma_{s,bot})\\ \mathbf{u}(0) &= 0,\quad \partial_t\mathbf{u}(0)=\mathbf{u}_{01} \text{ in } \Omega_s\end{aligned}$$

The fluid:

$$\begin{aligned}\partial_t\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= \mathbf{f} \text{ in } (0,T)\times\Omega_f\\ \text{div }\mathbf{v} &= 0 \text{ in } (0,T)\times\Omega_f\\ \mathbf{v}\times\mathbf{n} &= 0 \text{ on } (0,T)\times\Gamma_{f,ends}\\ \mathbf{v} &= 0 \text{ on } (0,T)\times\Gamma_{f,bot}\\ p &= 0 \text{ on } (0,T)\times\Gamma_{f,ends,2}\\ p &= P(t) \text{ on } (0,T)\times\Gamma_{f,ends,1}\\ \mathbf{v}(0) &= \mathbf{v}_0 \text{ in } \Omega_f,\end{aligned}$$

The coupling:

$$\begin{aligned}(\lambda\text{trace }\mathbf{e}(\mathbf{u})\mathbf{I} + 2\mu\mathbf{e}(\mathbf{u}))\cdot\mathbf{n}_s &= p\cdot\mathbf{n}_f - \nu(\nabla\times\mathbf{v})\times\mathbf{n}_f \text{ on } (0,T)\times\Gamma_{fs}\\ \partial_t\mathbf{u} &= \mathbf{v} \text{ on } (0,T)\times\Gamma_{fs}.\end{aligned}$$

Function spaces, notations, and assumptions

$$\mathcal{V} := \{\boldsymbol{\varphi} \in \mathcal{D}(\bar{\Omega}) \ : \ \text{div}\, \boldsymbol{\varphi} = 0 \text{ in } \Omega_f,\ \boldsymbol{\varphi} \times \mathbf{n} = 0 \text{ on } \Gamma_{f,ends},\ \boldsymbol{\varphi} = 0 \text{ on } \Gamma_{ext} \cup \Gamma_{s,ends} \cup \Gamma_{bot}\}$$

$$\mathbf{H}(\Omega) = \overline{\mathcal{V}}^{(\mathbf{L}^2(\Omega),(\cdot,\cdot)_{f,s})},\ \mathbf{V}(\Omega) = \overline{\mathcal{V}}^{\mathbf{H}^1(\Omega)},\ \mathbf{H}^{s,1}_{0,\Gamma} := \mathbf{H}^1_{0,\Gamma_{ext} \cup \Gamma_{s,ends} \cup \Gamma_{s,bot}}(\Omega_s).$$

$$\mathbf{V}_f := \{\mathbf{v} \in \mathbf{H}^1(\Omega_f) \ : \ \text{div}\, \mathbf{v} = 0 \text{ in } \Omega_f,\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_{f,ends},\ \mathbf{v} = 0 \text{ on } \Gamma_{f,bot}\}.$$

Denote by $(\boldsymbol{\xi}, \boldsymbol{\varphi})_{f,s}$ the $\mathbf{L}^2$-inner product

$$(\boldsymbol{\xi}, \boldsymbol{\varphi})_{f,s} := (\boldsymbol{\xi}, \boldsymbol{\varphi})_{\Omega_f} + (\boldsymbol{\xi}, \boldsymbol{\varphi})_{\Omega_s},\ \forall \boldsymbol{\xi}, \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega).$$

The norm in $\mathbf{L}^2(\Omega)$ is equivalent to the norm generated by this inner product.

Assume $\mathbf{g} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_s))$, $\mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_f))$ and
$P \in L^2(0,T;L^2(\Gamma_{f,ends,1}))$, $\mathbf{v}_0 \in \mathbf{V}_f$, $\mathbf{u}_{01} \in \mathbf{H}^{s,1}_{0,\Gamma}$ with $\mathbf{v}_0 = \mathbf{u}_{01}$ on Γ_{fs}.

Compatibility condition: $\int_{\Gamma_{f,ends,2}} v_3 - \int_{\Gamma_{f,ends,1}} v_3 + \int_{\Gamma_{fs}} \mathbf{v} \cdot \mathbf{n} = 0.$

One can prove that

$$\exists C_{curl} > 0 \ : \ \forall \mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{V}_f),\ \|\nabla \times \mathbf{v}\|^2_{\mathbf{L}^2(\Omega_f)} \geq C_{curl} ||\mathbf{v}||^2_{\mathbf{H}^1(\Omega_f)}.$$

The weak formulation

$\exists\, U_0 \in \mathbf{L}^2(0,T;\mathbf{V}_f)$ s.t.

$$\begin{aligned}
\text{div}\,\mathbf{U}_0 &= 0 \text{ in } (0,T)\times\Omega_f \\
\mathbf{U}_0 &= 0 \text{ on } (0,T)\times\Gamma_{f,bot} \\
\mathbf{U}_0 \times \mathbf{n} &= 0 \text{ on } (0,T)\times\Gamma_{f,ends}.
\end{aligned}$$

Problem 1: Find $(\mathbf{u},\mathbf{v}) \in \mathbf{L}^2(0,T;\mathbf{H}^{s,1}_{0,\Gamma}) \times \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))$ such that

$$\mathbf{v} - \mathbf{U}_0 \in \mathbf{L}^2(0,T;\mathbf{V}_f),$$

$$\frac{d}{dt}((\partial_t \mathbf{u},\boldsymbol{\varphi})_{\Omega_s} + (\mathbf{v},\boldsymbol{\varphi})_{\Omega_f}) + a(\mathbf{u},\boldsymbol{\varphi}) + \nu(\nabla\times\mathbf{v},\nabla\times\boldsymbol{\varphi})_{\Omega_f}$$

$$= (\mathbf{g},\boldsymbol{\varphi})_{\Omega_s} + (\mathbf{f},\boldsymbol{\varphi})_{\Omega_f} + \int\limits_{\Gamma_{f,ends,1}} P(t)\varphi_3,\ \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega),$$

where

$$a(\mathbf{u},\boldsymbol{\varphi}) := \lambda(\text{div}\,\mathbf{u}, \text{div}\,\boldsymbol{\varphi})_{\Omega_s} + 2\mu(\mathbf{e}(\mathbf{u}),\mathbf{e}(\boldsymbol{\varphi}))_{\Omega_s}$$

$$\mathbf{u}(0) = 0,\ \partial_t \mathbf{u}(0) = \mathbf{u}_{01},\ \mathbf{v}(0) = \mathbf{v}_0 \text{ and } \int\limits_0^t \mathbf{v}(s)ds = \mathbf{u}(t) \text{ a.e. } t \text{ on } \Gamma_{fs}.$$

An equivalent problem and the existence result

Notations: $\boldsymbol{\omega} := \partial_t \mathbf{u}\chi_{\Omega_s} + \mathbf{v}\chi_{\Omega_f}, \quad \boldsymbol{\omega}_0 := \mathbf{u}_{01}\chi_{\Omega_s} + \mathbf{v}_0\chi_{\Omega_f}, \quad \mathbf{G} := \mathbf{g}\chi_{\Omega_s} + \mathbf{f}\chi_{\Omega_f}.$

Problem 2: Find $\boldsymbol{\omega}$ such that $\mathbf{v} - \mathbf{U}_0 \in \mathbf{L}^2(0,T;\mathbf{V}_f)$ and

$$< \partial_t \boldsymbol{\omega}, \boldsymbol{\varphi} >_{f,s} + a(\int_0^t \boldsymbol{\omega}(s)ds, \boldsymbol{\varphi}) + \nu(\nabla \times \boldsymbol{\omega}, \nabla \times \boldsymbol{\varphi})_{\Omega_f}$$

$$= (\mathbf{G}(t), \boldsymbol{\varphi})_{f,s} + \int_{\Gamma_{f,ends,1}} P(t)\varphi_3, \ \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega) \text{ a.e. } t \in [0,T],$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0 \text{ in } \mathbf{V}'(\Omega),$$

$$\int_0^t \boldsymbol{\omega}(s)\chi_{\Omega_s} ds = \int_0^t \boldsymbol{\omega}(s)\chi_{\Omega_f} ds \text{ on } \Gamma_{fs}, \text{ a.e. } t,$$

where $\mathbf{V}'(\Omega)$ is the dual space of $\mathbf{V}(\Omega)$ and $< .,. >_{f,s}$ is the duality pairing between $\mathbf{V}'(\Omega)$ and $\mathbf{V}(\Omega)$, that is generated from the inner product $(.,.)_{f,s}$.

Theorem 1. *There exists a unique weak solution of our fluid-elastic structure coupling problem.*

Sketch of the proof

- Galerkin approximations, existence of a unique approximate solution $\boldsymbol{\omega}_m$
- Energy estimates:

Proposition 1. There exists a constant $C > 0$ such that

$$\sup_{0\le t\le T}\left(\|\boldsymbol{\omega}_m(t)\|^2_{f,s} + \|\int_0^t \boldsymbol{\omega}_m(s)ds\|^2_{\mathbf{H}^1(\Omega_s)}\right) + \|\boldsymbol{\omega}_m\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))} + \|\boldsymbol{\omega}'_m\|^2_{\mathbf{L}^2(0,T;\mathbf{V}'(\Omega))}$$

$$\le C(\|G\|^2_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|P\|^2_{L^2(0,T;L^2(\Gamma_{f,ends,1}))} + \|\mathbf{u}_{01}\|^2_{\mathbf{H}^1(\Omega_s))} + \|\mathbf{v}_0\|^2_{\mathbf{H}^1(\Omega_f)}).$$

The constant C depends on the fixed $T > 0$, on C_{curl}, ν and on the constants C_{trace} and C_{Korn} in the inequalities (with ζ in the corresponding spaces):

$$\|\zeta\|_{L^2(\Gamma_{f,ends,1})} \le C_{trace}\|\zeta\|_{H^1(\Omega_f)} \text{ (Sobolev embeddings)}$$

$$\|\zeta\|^2_{H^1(\Omega_s)} \le C_{Korn}a(\zeta,\zeta) \text{ (with Korn's inequality).}$$

- Passing to the limit and uniqueness proof.

Galerkin approximations

Consider the functions $\mathbf{w}_k = \mathbf{w}_k(\mathbf{x})$ ($k = 1, 2, ...$) s.t. $\{\mathbf{w}_k\}_k$ is a basis of $\mathbf{V}(\Omega)$.

Take $\{\mathbf{w}_k\}_k$ to be the complete set of eigenfunctions of the eigenvalue problem

$$\mathbf{w} \in \mathbf{V}(\Omega) \ : \ (\nabla \mathbf{w}, \nabla \boldsymbol{\varphi})_{f,s} = \alpha(\mathbf{w}, \boldsymbol{\varphi})_{f,s}, \ \forall \boldsymbol{\varphi} \in \mathbf{V}(\Omega).$$

Assume $\{\mathbf{w}_i\}_{i=1,2...}$ is orthonormalized with the $\mathbf{H}^s_{0,\Gamma}$ -inner product $(\nabla ., \nabla .)_{f,s}$. $\{\mathbf{w}_k\}_k$ is orthogonal w.r.t. the L^2-inner product $(., .)_{f,s}$.

$$\boldsymbol{\omega}_m(t) := \sum_{k=1}^{m} c_{km}(t)\mathbf{w}_k, \quad m \in \mathbb{N} \quad \text{(fixed)}.$$

with $c_{km}(t)$ ($0 \le t \le T$, $k = 1, \ldots, m$) s.t. $(\boldsymbol{\omega}_m(0), \mathbf{w}_k)_{f,s} = (\boldsymbol{\omega}_0, \mathbf{w}_k)_{f,s}$.

$$\begin{aligned}(\partial_t \boldsymbol{\omega}_m(t), \mathbf{w}_k)_{f,s} \quad &+ \quad a(\int_0^t \boldsymbol{\omega}_m(s)ds, \mathbf{w}_k) + \nu(\nabla \times \boldsymbol{\omega}_m(t), \nabla \times \mathbf{w}_k)_{\Omega_f} \\ &= \quad (\mathbf{G}(t), \mathbf{w}_k)_{f,s} + \int_{\Gamma_{f,ends,1}} P(t) w_{k,3}.\end{aligned}$$

The compatibility condition is obviously satisfied:

$$\int_0^t \boldsymbol{\omega}_m(s)\chi_{\Omega_s} ds = \int_0^t \boldsymbol{\omega}_m(s)\chi_{\Omega_f} ds \text{ on } \Gamma_{fs}, \text{ a.e. } t.$$

Galerkin approximations

Denote the right-hand side in the Galerkin approximations by $F_k(t)$ and observe that the above system can be written in the form of a linear ODE system of first order for $c_{km}(t)$ and for $d_{km}(t) := \int\limits_0^t c_{km}(s)ds$:

$$\sum_{l=1}^{m}(\mathbf{w}_l,\mathbf{w}_k)_{f,s}c'_{lm}(t)+\nu\sum_{l=1}^{m}(\nabla\times\mathbf{w}_l,\nabla\times\mathbf{w}_k)_{\Omega_f}c_{lm}(t)+\sum_{l=1}^{m}a(\mathbf{w}_l,\mathbf{w}_k)d_{lm}(t)=F_k(t),$$

with

$$d'_{lm}(t)=c_{lm}(t),\ l=1,\ldots,m$$

and with the initial conditions

$$\begin{aligned}\sum_{l=1}^{m}(\mathbf{w}_l,\mathbf{w}_k)_{f,s}c_{lm}(0) &= (\boldsymbol{\omega}_0,\mathbf{w}_k)_{f,s},\\ d_{lm}(0) &= 0,\ l=1,\ldots,m.\end{aligned}$$

This leads to the existence of a unique solution $\boldsymbol{\omega}_m$ to the previous approximating system.

Strong energy estimates

Theorem 2. Let $\mathbf{g}$, $\mathbf{f}$, P, $\mathbf{v}_0$ and $\mathbf{u}_{01}$ as before and assume

$$\mathbf{g}' \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_s)),\ \mathbf{f}' \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_f)),\ P' \in L^2(0,T;L^2(\Gamma_{f,ends,1})),$$

$\mathbf{v}_0 \in \mathbf{H}^2(\Omega_f)$ with $\mathbf{v}_0 \times \mathbf{n} = 0$ on $\Gamma_{f,ends}$, $\mathbf{u}_1 \in \mathbf{H}^2(\Omega_s)$, $P(0) \in L^2(\Gamma_{f,ends,1})$.

Also assume $\exists\, p_0 \in H^1(\Omega_f)$ such that
$p_0 \cdot \mathbf{n}_f - \nu(\nabla \times \mathbf{v}_0) \times \mathbf{n}_f = 0$ on $(0,T) \times \Gamma_{fs}$.

Then the following estimate holds:

$$\|\mathbf{v}'(t)\|^2_{\mathbf{L}^2(\Omega_f)} + \|\mathbf{u}''(t)\|^2_{\mathbf{L}^2(\Omega_s)} + \|\mathbf{v}'\|^2_{\mathbf{L}^2(0,T;\mathbf{H}^1_{0,\Gamma_{f,bot}}(\Omega_f))} + \|\mathbf{u}'(t)\|^2_{\mathbf{H}^1(\Omega_s)}$$

$$\le Ce^{CT}\Big(\|\mathbf{G}\|^2_{\mathbf{H}^1(0,T;\mathbf{L}^2(\Omega))} + \|P'\|^2_{L^2(0,T;L^2(\Gamma_{f,ends,1}))} + \|\mathbf{v}_0\|^2_{\mathbf{H}^2(\Omega_f)}$$

$$+\|p_0\|^2_{H^1(\Omega_f)} + \|\mathbf{u}_{01}\|^2_{\mathbf{H}^1(\Omega_s)}\Big).$$

Moreover, for $\mathbf{U}_0 \in \mathbf{L}^2(0,T;\mathbf{V}_f)$, $\mathbf{v}$ satisfies div $\mathbf{v} = 0$ in the sense of distributions in Ω and the boundary conditions on $\mathbf{v}$ in the sense of traces of functions of $\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_f))$.

Existence of a pressure

Theorem 3. Assume $\nabla \cdot \mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_f))$. Then in the hypotheses of Theorem 2 there exists a class of functions $p \in H(\Delta,\Omega_f)/\mathbb{R}$ such that

$$\partial_t \mathbf{v} - \nu \mathbf{\Delta v} + \nabla p = \mathbf{f}$$

is satisfied in the sense of distributions in Ω_f.

Moreover, for $\nabla \times \mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_f))$ we have $\nabla \times \mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}(\Delta,\Omega_f))$.

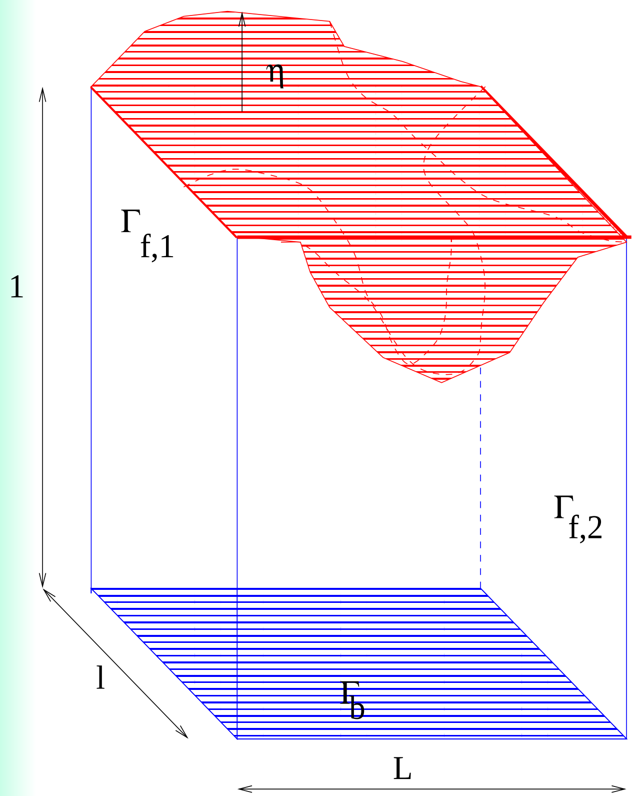
Here

$$H(\Delta,\Omega_f) := \{q \in L^2(\Omega_f) \ : \ \Delta q \in L^2(\Omega_f)\}.$$

and $p \in H(\Delta,\Omega_f)/\mathbb{R}$ means that $p \in H(\Delta,\Omega_f)$ and it is unique up to an additive constant.

What about a Navier-Stokes fluid in a time-dependent domain?

A result in the time-dependent case (noncyl. domain)



$\Omega_s = (0, L) \times (0, l)$, η_0 given initial displacement

$$\Omega_{\eta_0} := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : (x_1, x_2) \in \Omega_s, \; 0 < x_3 < 1 + \eta_0(x_1, x_2)\} \text{ (initial state)}$$

$$\Omega_\eta(t) := \{(x_1, x_2, x_3) \in \mathbf{R}^3 : (x_1, x_2) \in \Omega_s, \; 0 < x_3 < 1 + \eta(t, x_1, x_2)\} \text{ (at time t)}$$

Boundaries:

$\Gamma_{f,1}$, $\Gamma_{f,2}$ are the inflow, respectively the outflow boundaries.
Γ_b is the (fixed) bottom boundary of the fluid-filled box.
$\Gamma_{sides} = \Gamma_{front} \cup \Gamma_{back}$ are the side boundaries of the fluid domain (fixed).
Γ_{fs} is the time moving fluid-structure interface.
$\partial[(0, L) \times (0, l) \times \{1\}]$ is the (clamped) boundary of the elastic plate.

Aim: does a solution of the coupled problem exist?

The equations

For the elastic plate (only transversal displacement):

$$\begin{aligned}\partial_{tt}\eta + \Delta^2\eta + \gamma\Delta^2\partial_t\eta &= g + G \text{ in } (0,T)\times\Omega_s \\ \eta = \partial_n\eta &= 0 \text{ on } (0,T)\times\partial\Omega_s \\ \eta(0) &= \eta_0,\ \partial_t\eta(0) = \eta_{01} \text{ in } \Omega_s,\end{aligned}$$

where $G{:=}(F_f)_3$ and $\mathbf{F}_f$ is the force applied by the fluid on the structure.

For the fluid:

$$\begin{aligned}\partial_t\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \text{ in } Q_{\eta,T} \\ \text{div } \mathbf{u} &= 0 \text{ in } Q_{\eta,T} \\ \mathbf{u}\times\mathbf{n} &= 0 \text{ on } (0,T)\times\Gamma_f \\ \mathbf{u} &= 0 \text{ on } (0,T)\times(\Gamma_b\cup\Gamma_{sides}) \\ p &= p_{0i} \text{ on } (0,T)\times\Gamma_{f,i}\ (i=1,2) \\ \mathbf{u}(0) &= \mathbf{u}_0 \text{ in } \Omega_{\eta_0},\end{aligned}$$

where $Q_{\eta,T}\subset\mathbf{R}^4$ is defined as $Q_{\eta,T} := \bigcup_{0<t<T}\{t\}\times\Omega_\eta(t)$.

The coupling

Equality of velocities at the interface:

$$\mathbf{u}(t, x_1, x_2, 1 + \eta(t, x_1, x_2)) = (0, 0, \partial_t \eta(t, x_1, x_2)), \ (x_1, x_2) \in \Omega_s.$$

Compatibility condition:

$$\int_{\Omega_s} \partial_t \eta - \int_{\Gamma_{f1}} u_1 + \int_{\Gamma_{f2}} u_1 = 0.$$

The surface force exerted by the fluid on the elastic wall:

$$\int_{\Omega_s} \mathbf{F}_f \cdot \tilde{\mathbf{v}} = \int_{\partial\Omega_\eta(t) - (\Gamma_f \cup \Gamma_b \cup \Gamma_{sides})} (-\nu(\nabla \times \mathbf{u}) \times \mathbf{n}_t + p \cdot \mathbf{n}_t) \cdot \mathbf{v}, \ \forall \mathbf{v},$$

where $\tilde{\mathbf{v}}(t, x_1, x_2) = \mathbf{v}(t, x_1, x_2, 1 + \eta(t, x_1, x_2)), \forall (x_1, x_2) \in \Omega_s$;

$\mathbf{n}_t$ is the unit outer normal at $\Gamma_{fs}(t) := \partial\Omega_\eta(t) - (\Gamma_f \cup \Gamma_b \cup \Gamma_{sides})$.

Observe that

$$d\Gamma_{fs}(t) = \sqrt{1 + (\partial_{x_1}\eta)^2 + (\partial_{x_2}\eta)^2}\, dx_1 dx_2.$$

A priori estimates

Proposition 1. Assume that $\mathbf{u}_0 \in \mathbf{L}^2(\Omega_{\eta_0})$, $\eta_{01} \in L^2(\Omega_s)$, $\eta_0 \in H_0^2(\Omega_s)$, $\mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))$, $p_0 \in L^2(0,T;L^2(\Gamma_f))$ and $g \in L^2(0,T;L^2(\Omega_s))$.
Then an a priori estimate holds, so that one gets

$$\mathbf{u} \in \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\eta(t))) \cap \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\eta(t)))$$

and

$$\eta \in W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H_0^2(\Omega_s)) \cap L^\infty(0,T;H_0^2(\Omega_s)).$$

Proof: use the Reynolds transport formula, the usual Sobolev embeddings, Gronwall's inequality and the following *ellipticity condition:*

$$\exists\, c_E > 0 \textit{ s.t. } \forall \mathbf{u} \in \mathbf{V} \quad |\nabla \times \mathbf{u}(t)|^2_{\mathbf{L}^2(\mathcal{B}_K)} \geq c_E ||\mathbf{u}(t)||^2_{\mathbf{H}^1(\mathcal{B}_K)},$$

where $\mathcal{B}_K := \Omega_s \times (0,K)$ *and*

$$\begin{aligned} \mathbf{V} \quad := \quad & \{\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_K)) \;:\; \mathit{div}\,\mathbf{v} = 0,\; \mathbf{v} = 0 \textit{ on } (0,T) \times (\Gamma_b \cup \Gamma_{sides}), \\ & \mathbf{v} \times \mathbf{n} = 0 \textit{ on } (0,T) \times \Gamma_f\} \end{aligned}$$

Some tools: function spaces

Let $T > 0$, $\delta \in H^1(0,T;H_0^2(\Omega_s))$; take $K \geq 1 + \delta(t,x_1,x_2) \geq \alpha > 0$ in $[0,T] \times \bar{\Omega}_s$.

For every $t \in [0,T]$, $\Omega_\delta(t) \subset \mathcal{B}_K$. Define $\mathcal{B}_{K,T} := (0,T) \times \mathcal{B}_K$.

As usual, $L^q(\Omega_\delta(t))$, $H^1(\Omega_\delta(t))$ ($\forall t$), $L^q(Q_{\delta,T})$, $H^1(Q_{\delta,T})$, $L^q(\mathcal{B}_{K,T})$, $H^1(\mathcal{B}_{K,T})$, ...

$$\begin{aligned} \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t))) &:= \{\mathbf{v} \in \mathbf{L}^2(Q_{\delta,T}) \,:\, \nabla\mathbf{v} \in \mathbf{L}^2(Q_{\delta,T})\}, \\ \mathbf{L}^2(0,T;\mathbf{H}_0^1(\Omega_\delta(t))) &:= \overline{\mathcal{D}(Q_{\delta,T})}^{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t)))}, \end{aligned}$$

$$\begin{aligned} \mathcal{V}_\delta := \{&\mathbf{v} \in \mathbf{C}^1(\bar{Q}_{\delta,T}) \,:\, \text{div}\,\mathbf{v} = 0,\ \mathbf{v} = 0 \text{ on } (0,T) \times (\Gamma_b \cup \Gamma_{sides}), \\ &\mathbf{v} \times \mathbf{n} = 0 \text{ on } (0,T) \times \Gamma_f\}, \end{aligned}$$

$$\begin{aligned} \mathbf{V}_\delta &:= \overline{\mathcal{V}_\delta}^{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t)))}, \\ \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\delta(t))) &:= \{\mathbf{v} \in \mathbf{L}^2(Q_{\delta,T}) \,:\, \text{sup ess}_{0<t<T} ||\mathbf{v}||_{\mathbf{L}^2(\Omega_\delta(t))} < \infty\}. \end{aligned}$$

$$\begin{aligned} \mathbf{V}_\delta = \{&\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\delta(t))) \,:\, \text{div}\,\mathbf{v} = 0,\ \mathbf{v} = 0 \text{ on } (0,T) \times (\Gamma_b \cup \Gamma_{sides}), \\ &\mathbf{v} \times \mathbf{n} = 0 \text{ on } (0,T) \times \Gamma_f\}. \end{aligned}$$

Further tools: some lemmas

Consider the mapping $\mathbf{C}^0(\overline{\Omega_\delta}(t)) \ni \mathbf{v} \overset{\gamma_{\delta(t)}}{\mapsto} \mathbf{v}(t, x_1, x_2, 1 + \delta(t, x_1, x_2))$.

Lemma 1. (trace on $\partial\Omega_\delta(t) - (\Gamma_b \cup \Gamma_{sides} \cup \Gamma_f)$)
For every $t \in [0, T]$, the mapping $\gamma_{\delta(t)} : \mathbf{C}^1(\overline{\mathcal{B}}_K)$ (respectively $\mathbf{C}^1(\bar{\Omega}_\delta(t))$) $\to \mathbf{C}^0(\bar{\Omega}_s)$ can be extended by continuity to a mapping from $\mathbf{H}^1(\mathcal{B}_K)$ (respectively $\mathbf{H}^1(\Omega_\delta(t))$) into $\mathbf{L}^2(\Omega_s)$.

Lemma 2. For every $t \in [0, T]$, there exists a linear continuous operator $\gamma^{tg}_{\delta(t)} : \mathbf{H}(\mathbf{curl}, \Omega_\delta(t)) \to \mathbf{H}^{-1}(\Omega_s)$ such that

$$\gamma^{tg}_{\delta(t)}(\mathbf{v}) = \mathbf{v}(t, x_1, x_2, 1 + \delta(t, x_1, x_2)) \times \mathbf{n}_t, \ \forall (x_1, x_2) \in \Omega_s,$$

for all $\mathbf{v} \in \mathbf{C}^\infty(\bar{\Omega}_\delta(t))$, where

$$\mathbf{H}(\mathbf{curl}, \Omega_\delta(t)) := \{\mathbf{v} \in \mathbf{L}^2(\Omega_\delta(t)) : \mathbf{curl}\ \mathbf{v} \in \mathbf{L}^2(\Omega_\delta(t))\}.$$

Further tools: some lemmas

Lemma 3. For every $t \in [0,T]$, there exists a linear continuous operator $\gamma^n_{\delta(t)} : \mathbf{H}(div, \Omega_\delta(t)) \to H^{-1}(\Omega_s)$ such that

$$\gamma^n_{\delta(t)}(\mathbf{v}) = \mathbf{v}(t, x_1, x_2, 1 + \delta(t, x_1, x_2)) \cdot \mathbf{n}_t, \ \forall (x_1, x_2) \in \Omega_s,$$

for all $\mathbf{v} \in \mathbf{C}^\infty(\bar{\Omega}_\delta(t))$, where

$$\mathbf{H}(div, \Omega_\delta(t)) := \{\mathbf{v} \in \mathbf{L}^2(\Omega_\delta(t)) : \ div\, \mathbf{v} \in \mathbf{L}^2(\Omega_\delta(t))\}.$$

Lemma 4.

$$\{\mathbf{v} \in \mathbf{H}^1_{0,\Gamma_b \cup \Gamma_{sides} \cup \Gamma_1(t)}(\Omega_\delta(t)) \ : \ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_f\}$$

$$= \{\mathbf{v} \in \mathbf{H}^1_{0,\Gamma_b \cup \Gamma_{sides}}(\Omega_\delta(t)) \ : \ \gamma_{\delta(t)}(\mathbf{v}) = 0, \ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_f\}.$$

Lemma 5. Let $\mathbf{v} \in \mathbf{V}_\delta$ s.t. for a.e. t, $\gamma_{\delta(t)}(\mathbf{v}) = (0,0,b)$, with $b \in L^2(0,T; H^1_0(\Omega_s))$.
Then define the function

$$\tilde{\mathbf{v}} := \begin{cases} \mathbf{v} & \text{in} \quad Q_{\delta,T} \\ (0,0,b) & \text{in} \quad \mathcal{B}_{K,T} - Q_{\delta,T} \end{cases}.$$

$\tilde{\mathbf{v}}$ belongs to $\mathbf{V}$ and $||\tilde{\mathbf{v}}||_{\mathbf{V}} \leq C(||\mathbf{v}||_{\mathbf{V}_\delta} + ||b||_{L^2(0,T;H^1_0(\Omega_s))})$.

Weak formulation

Definition. $(\mathbf{u}, \eta)$ is a weak solution of the coupled problem (on $[0,T)$) if:

- $\mathbf{u} \in \mathbf{V}_\eta \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\eta(t)))$, $\eta \in W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H_0^2(\Omega_s))$
- $\gamma_{\eta(t)}(\mathbf{u}) = (0,0,\partial_t \eta)$ for a.e. t
- for all $(\boldsymbol{\psi}, b) \in \mathcal{V}_\eta \times C^1(0,T;H_0^2(\Omega_s))$ s.t.

$$\boldsymbol{\psi}(t,x_1,x_2,1+\eta(t,x_1,x_2)) = (0,0,b(t,x_1,x_2)), (t,x_1,x_2) \in [0,T] \times \Omega_s$$

we have for a.e. t

$$\int_{\Omega_\eta(t)} \mathbf{u}(t)\cdot\boldsymbol{\psi}(t) - \int_0^t \int_{\Omega_\eta(s)} \mathbf{u}\cdot\partial_t\boldsymbol{\psi} + \nu \int_0^t \int_{\Omega_\eta(s)} (\nabla\times\mathbf{u})\cdot(\nabla\times\boldsymbol{\psi}) + \int_0^t \int_{\Omega_\eta(s)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\psi}$$

$$-\int_0^t \int_{\Omega_s} \partial_t\eta\partial_t b + \int_{\Omega_s} \partial_t\eta(t) b(t) + \int_0^t \int_{\Omega_s} \Delta\eta\Delta b - \int_0^t \int_{\Omega_s} (\partial_t\eta)^2 b + \gamma \int_0^t \int_{\Omega_s} \Delta\partial_t\eta\Delta b$$

$$= \int_0^t \int_{\Omega_\eta(s)} \mathbf{f}\cdot\boldsymbol{\psi} + \int_0^t \int_{\Omega_s} g\cdot b - \int_0^t \int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\boldsymbol{\psi} + \int_{\Omega_{\eta 0}} \mathbf{u}_0\cdot\boldsymbol{\psi}(0) + \int_{\Omega_s} \eta_{01} b(0).$$

The main result

Let $\eta_0 \in H^2_0(\Omega_s)$, $\mathbf{u}_0 \in \mathbf{L}^2(\Omega_{\eta_0})$, $p_0 \in L^2(0,T;H^{1/2}(\Gamma_f))$, $\eta_{01} \in L^2(\Omega_s)$ s.t.

$\min_{\bar{\Omega}_s}(1+\eta_0) > 0$, div $\mathbf{u}_0 = 0$, $\mathbf{u}_0 = 0$ on $\Gamma_b \cup \Gamma_{sides}$, $\mathbf{u}_0 \times \mathbf{n} = 0$ on Γ_f,

$\gamma^n_{\eta_0}(\mathbf{u}_0) = (0,0,\eta_{01}) \cdot \mathbf{n}_0$ on Ω_s, $\int_{\Omega_s} \eta_{01} - \int_{\Gamma_{f2}} u_1(0) + \int_{\Gamma_{f1}} u_1(0) = 0$.

THEOREM. Let $\mathbf{f} \in \mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))$, $g \in L^2(0,T;L^2(\Omega_s))$.
Then there exists $T>0$ and a weak solution of the problem on $[0,T]$, which satisfies the following estimates:

$$||\mathbf{u}||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_\eta(t)))} + ||\mathbf{u}||_{\mathbf{L}^2(0,T;\mathbf{H}^1(\Omega_\eta(t)))} + ||\partial_t \eta||_{L^\infty(0,T;L^2(\Omega_s))} + ||\Delta\eta||_{H^1(0,T;L^2(\Omega_s))}$$

$$\leq \text{const}\,(T, ||\mathbf{u}_0||_{\mathbf{L}^2(\Omega_{\eta_0})},\ ||\mathbf{f}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))}, ||p_0||_{L^2(0,T;L^2(\Gamma_f))},$$

$$||g||_{L^2(0,T;L^2(\Omega_s))},\ ||\eta_0||_{H^2_0(\Omega_s)},\ ||\eta_{01}||_{L^2(\Omega_s)}).$$

Idea of the proof

- State an approximate problem, whose solutions are built by regularizing the convection velocities.
- Apply Schauder's Second Fixed-Point Theorem to show the existence of a solution for the approximate problem.
- Prove some compactness results.
- Pass to the limit in the approximate problem.

Remark. (rewriting the convective term)

$$\int_{\Omega_\eta(t)} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\psi} = \int_{\Omega_\eta(t)} (\nabla\times\mathbf{u})\times\mathbf{u}\cdot\boldsymbol{\psi} + \frac{1}{2}\int_{\Omega_s}(\partial_t\eta)^2 b - \frac{1}{2}\int_{\Gamma_{f1}} u_1^2\psi_1 + \frac{1}{2}\int_{\Gamma_{f2}} u_1^2\psi_1 .$$

Let $\mathbf{u}_0^\epsilon$, η_0^ϵ, η_{01}^ϵ be regularizations of the initial data s.t.

- div $\mathbf{u}_0^\epsilon = 0$,
- $\mathbf{u}_0^\epsilon(x_1, x_2, 1+\eta_0^\epsilon(x_1,x_2)) = (0,0,\eta_{01}^\epsilon(x_1,x_2))$ in Ω_s,
- $\mathbf{u}_0^\epsilon = 0$ on $\Gamma_b\cup\Gamma_{sides}$, $\mathbf{u}_0^\epsilon\times\mathbf{n} = 0$ on Γ_f,
- $\int_{\Omega_s}\eta_{01}^\epsilon - \int_{\Gamma_{f1}} u_{0,1}^\epsilon + \int_{\Gamma_{f2}} u_{0,1}^\epsilon = 0$
- $\chi_{\Omega_{\eta_0{}^\epsilon}}\mathbf{u}_0^\epsilon \overset{\epsilon\to 0}{\rightarrow} \chi_{\Omega_{\eta_0}}\mathbf{u}_0$ in $\mathbf{L}^2(\mathcal{B}_K)$
- $\eta_{01}^\epsilon \overset{\epsilon\to 0}{\rightarrow} \eta_{01}$ in $L^2(\Omega_s)$ and $\eta_0^\epsilon \overset{\epsilon\to 0}{\rightarrow} \eta_0$ in $H_0^2(\Omega_s)$.

Using these regularizations, construct a sequence $(\mathbf{u}_\epsilon, \eta_\epsilon)_{\epsilon>0}$ of approximate weak solutions.

The approximate problem

Proposition 2. Let $\tilde{\mathbf{u}}_\epsilon^\sharp$ and $\eta_\epsilon^\sharp$ be regularizations of $\tilde{\mathbf{u}}_\epsilon$, respectively η_ϵ. Then there exists $(\mathbf{u}_\epsilon, \eta_\epsilon) \in (\mathbf{V}_{\eta_\epsilon^\sharp} \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t)))) \times (W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H_0^2(\Omega_s)))$

- $\mathbf{u}_\epsilon(t,x_1,x_2,1+\eta_\epsilon^\sharp(t,x_1,x_2)) = (0,0,\partial_t\eta_\epsilon(t,x_1,x_2))$ on Ω_s,
- $\partial_t\mathbf{u}_\epsilon \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t)))$, $\partial_{tt}\eta_\epsilon \in L^2(0,T;L^2(\Omega_s))$,
- $\mathbf{u}_\epsilon(0) = \mathbf{u}_0^\epsilon$, $\eta_\epsilon(0) = \eta_0^\epsilon$, $\partial_t\eta_\epsilon(0) = \eta_{01}^\epsilon$ and

$$\int_0^t \int_{\Omega_{\eta_\epsilon^\sharp}(s)} \partial_t\mathbf{u}_\epsilon \cdot \boldsymbol{\psi}_\epsilon + \nu \int_0^t \int_{\Omega_{\eta_\epsilon^\sharp}(s)} (\nabla\times\mathbf{u}_\epsilon)\cdot(\nabla\times\boldsymbol{\psi}_\epsilon) + \int_0^t \int_{\Omega_{\eta_\epsilon^\sharp}(s)} (\nabla\times\tilde{\mathbf{u}}_\epsilon^\sharp)\times\mathbf{u}_\epsilon\cdot\boldsymbol{\psi}_\epsilon$$
$$-\frac{1}{2}\int_0^t \int_{\Gamma_{f1}} u_{\epsilon,1}\tilde{u}_{\epsilon,1}^\sharp\psi_{\epsilon,1} + \frac{1}{2}\int_0^t \int_{\Gamma_{f2}} u_{\epsilon,1}\tilde{u}_{\epsilon,1}^\sharp\psi_{\epsilon,1} + \frac{1}{2}\int_0^t \int_{\Omega_s} \partial_t\eta_\epsilon\partial_t\eta_\epsilon^\sharp b + \int_0^t \int_{\Omega_s} \partial_{tt}\eta_\epsilon b$$
$$+\int_0^t \int_{\Omega_s} \Delta\eta_\epsilon\Delta b + \gamma\int_0^t \int_{\Omega_s} \Delta(\partial_t\eta_\epsilon)\Delta b = \int_0^t \int_{\Omega_{\eta_\epsilon^\sharp}(s)} \mathbf{f}\cdot\boldsymbol{\psi}_\epsilon - \int_0^t \int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\boldsymbol{\psi}_\epsilon + \int_0^t \int_{\Omega_s} g\cdot b,$$

$\forall \boldsymbol{\psi}_\epsilon \in \mathbf{V}_{\eta_\epsilon^\sharp}$, $b \in L^2(0,T;H_0^2(\Omega_s))$ such that

$$\boldsymbol{\psi}_\epsilon(t,x_1,x_2,1+\eta_\epsilon^\sharp(t,x_1,x_2)) = (0,0,b(t,x_1,x_2)) \text{ on } \Omega_s.$$

Idea of the proof for Proposition 2

- Step 1:
  - linearize the weak formulation of the approximate problem
  - transform to the reference configuration and use the method of Galerkin to show existence of a unique solution to the linearized approximate problem
- Step 2: apply Schauder's Generalized Fixed Point Theorem to show existence of a weak solution $(\mathbf{u}_\epsilon, \eta_\epsilon)$ to the approximate problem

The approximate and linearized problem

Let $\delta \in H^1(0,T;H_0^2(\Omega_s))$, $\delta(0) = \eta_0^\epsilon$ and $K \geq 1 + \delta(t,x_1,x_2) \geq \alpha > 0$, $\forall (t,x_1,x_2) \in [0,T] \times \bar{\Omega}_s$ (α is such that $\min_{\bar{\Omega}_s}(1+\eta_0) \geq 2\alpha > 0$).
Take $\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$.

Consider $\delta_\epsilon^\sharp = R_\epsilon^s(\delta)$ and $\mathbf{v}_\epsilon^\sharp = R_\epsilon^f(\mathbf{v})$ space-time regularizations of δ, respectively $\mathbf{v}$, such that:

$$R_\epsilon^s(\delta)|_{t=0} = \eta_0^\epsilon \text{ and } 2K \geq 1 + \delta_\epsilon^\sharp(t,x_1,x_2) \geq \frac{\alpha}{2},\ \forall (t,x_1,x_2) \in [0,T] \times \bar{\Omega}_s.$$

$$R_\epsilon^s(\delta_\epsilon) \to \delta \text{ in } C([0,T] \times \bar{\Omega}_s) \text{ when } \delta_\epsilon \to \delta \text{ in } C([0,T] \times \bar{\Omega}_s),$$

$$\partial_t R_\epsilon^s(\delta_\epsilon) \to \partial_t \delta \text{ in } L^2(0,T;L^2(\Omega_s)) \text{ when } \partial_t \delta_\epsilon \to \partial_t \delta \text{ in } L^2(0,T;L^2(\Omega_s))$$

$$R_\epsilon^f(\mathbf{v}_\epsilon) \to \mathbf{v} \text{ in } \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K})) \text{ when } \mathbf{v}_\epsilon \text{ converges to } \mathbf{v} \text{ in } \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K})).$$

The approximate and linearized problem

Find $(\mathbf{u}_\epsilon, \eta_\epsilon)$ such that:

- $\mathbf{u}_\epsilon \in \mathbf{V}_{\delta^\sharp_\epsilon} \cap \mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_{\delta^\sharp_\epsilon}(t)))$,
- $\eta_\epsilon \in W^{1,\infty}(0,T;L^2(\Omega_s)) \cap H^1(0,T;H^2_0(\Omega_s))$,
- $\partial_t \mathbf{u}_\epsilon \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\delta^\sharp_\epsilon}(t)))$, $\partial_{tt}\eta_\epsilon \in \mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_s))$,
- $\mathbf{u}_\epsilon(t,x_1,x_2,1+\delta^\sharp_\epsilon(t,x_1,x_2)) = (0,0,\partial_t\eta_\epsilon(t,x_1,x_2))$ on Ω_s,
- $\mathbf{u}_\epsilon(0) = \mathbf{u}^\epsilon_0$, $\eta_\epsilon(0) = \eta^\epsilon_0$, $\partial_t\eta_\epsilon(0) = \eta^\epsilon_{01}$ and

$$\int_0^t \int_{\Omega_{\delta^\sharp_\epsilon}(s)} \partial_t \mathbf{u}_\epsilon \cdot \boldsymbol{\psi}_\epsilon + \nu \int_0^t \int_{\Omega_{\delta^\sharp_\epsilon}(s)} (\nabla \times \mathbf{u}_\epsilon)\cdot(\nabla\times\boldsymbol{\psi}_\epsilon) + \int_0^t \int_{\Omega_{\delta^\sharp_\epsilon}(s)} (\nabla\times\mathbf{v}^\sharp_\epsilon)\times\mathbf{u}_\epsilon\cdot\boldsymbol{\psi}_\epsilon$$

$$-\frac{1}{2}\int_0^t\int_{\Gamma_{f1}} u_{\epsilon,1}v^\sharp_{\epsilon,1}\cdot\psi_{\epsilon,1} + \frac{1}{2}\int_0^t\int_{\Gamma_{f2}} u_{\epsilon,1}v^\sharp_{\epsilon,1}\cdot\psi_{\epsilon,1} + \frac{1}{2}\int_0^t\int_{\Omega_s}\partial_t\eta_\epsilon\partial_t\delta^\sharp_\epsilon b + \int_0^t\int_{\Omega_s}\partial_{tt}\eta_\epsilon b$$

$$+\int_0^t\int_{\Omega_s}\Delta\eta_\epsilon\Delta b + \gamma\int_0^t\int_{\Omega_s}\Delta(\partial_t\eta_\epsilon)\Delta b = \int_0^t\int_{\Omega_{\delta^\sharp_\epsilon}(s)}\mathbf{f}\cdot\boldsymbol{\psi}_\epsilon - \int_0^t\int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\boldsymbol{\psi}_\epsilon + \int_0^t\int_{\Omega_s} g\cdot b,$$

$\forall \boldsymbol{\psi}_\epsilon \in \mathbf{V}_{\delta^\sharp_\epsilon},\ b \in L^2(0,T;H^2_0(\Omega_s))$ s.t.

$$\boldsymbol{\psi}_\epsilon(t,x_1,x_2,1+\delta^\sharp_\epsilon(t,x_1,x_2)) = (0,0,b(t,x_1,x_2)) \text{ on } \Omega_s.$$

Transforming to the reference configuration

We denote the reference configuration by $R := \Omega_s \times (0,1)$ and consider the transformation

$$\boldsymbol{\phi}_\epsilon : (0,T) \times R \to \Omega_{\delta_\epsilon^\sharp}(t),$$

$$\boldsymbol{\phi}_\epsilon(t, x_1, x_2, x_3) := (x_1, x_2, x_3(1 + \delta_\epsilon^\sharp(t, x_1, x_2))),\ \forall (x_1, x_2, x_3) \in R,\ t \in (0,T).$$

Observe that

- $\boldsymbol{\phi}_\epsilon$ is smooth in space and time and
- $\partial_t \boldsymbol{\phi}_\epsilon = (0, 0, x_3 \partial_t \delta_\epsilon^\sharp)$.

Denote

$$\mathbf{u}_\epsilon^{\phi_\epsilon} := \mathbf{u}_\epsilon \circ \boldsymbol{\phi}_\epsilon,\ \boldsymbol{\psi}_\epsilon^{\phi_\epsilon} := \boldsymbol{\psi}_\epsilon \circ \boldsymbol{\phi}_\epsilon,\ \mathbf{f}^{\phi_\epsilon} := \mathbf{f} \circ \boldsymbol{\phi}_\epsilon,\ p_0^{\phi_\epsilon} := p_0 \circ \boldsymbol{\phi}_\epsilon$$

$$J_\epsilon := \det \nabla \boldsymbol{\phi}_\epsilon,\ \mathbf{M}_\epsilon := \operatorname{cof} \nabla \boldsymbol{\phi}_\epsilon,\ \mathbf{n}^{\phi_\epsilon} := \frac{\mathbf{M}_\epsilon \cdot \mathbf{n}}{||\mathbf{M}_\epsilon \cdot \mathbf{n}||},\ d\sigma^{\phi_\epsilon} = ||\mathbf{M}_\epsilon \cdot \mathbf{n}|| d\sigma$$

Transforming to the reference configuration

The equations become ($\mathbf{u}_\epsilon^{\phi_\epsilon}(t,x_1,x_2,1) = (0,0,\partial_t\eta_\epsilon(t,x_1,x_2)),\ (x_1,x_2)\in\Omega_s$):

$$\int_0^t\int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon}\boldsymbol{\psi}_\epsilon^{\phi_\epsilon} J_\epsilon + \nu\int_0^t\int_R ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla)\times\mathbf{u}_\epsilon^{\phi_\epsilon})((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla)\times\boldsymbol{\psi}_\epsilon^{\phi_\epsilon})$$

$$+\int_0^t\int_R ((\mathbf{M}_\epsilon\nabla)\times\mathbf{v}_\epsilon^{\phi_\epsilon,\sharp})\times\mathbf{u}_\epsilon^{\phi_\epsilon}\cdot\boldsymbol{\psi}_\epsilon^{\phi_\epsilon} - \int_0^t\int_R(\partial_t\boldsymbol{\phi}_\epsilon\cdot(\mathbf{M}_\epsilon\nabla))\mathbf{u}_\epsilon^{\phi_\epsilon}\cdot\boldsymbol{\psi}_\epsilon^{\phi_\epsilon} + \frac{1}{2}\int_0^t\int_{\Omega_s}\partial_t\eta_\epsilon\partial_t\delta_\epsilon^\sharp b$$

$$-\frac{1}{2}\int_0^t\int_{\Gamma_{f1}} u_{\epsilon,1}^{\phi_\epsilon}v_{\epsilon,1}^{\phi_\epsilon,\sharp}\cdot\psi_{\epsilon,1}^{\phi_\epsilon}\cdot J_\epsilon + \frac{1}{2}\int_0^t\int_{\Gamma_{f2}} u_{\epsilon,1}^{\phi_\epsilon}v_{\epsilon,1}^{\phi_\epsilon,\sharp}\cdot\psi_{\epsilon,1}^{\phi_\epsilon}\cdot J_\epsilon + \int_0^t\int_{\Omega_s}\partial_{tt}\eta_\epsilon b + \int_0^t\int_{\Omega_s}\Delta\eta_\epsilon\Delta b$$

$$+\gamma\int_0^t\int_{\Omega_s}\Delta(\partial_t\eta_\epsilon)\Delta b = \int_0^t\int_R \mathbf{f}^{\phi_\epsilon}\cdot\boldsymbol{\psi}_\epsilon^{\phi_\epsilon}J_\epsilon + \int_0^t\int_{\Omega_s} g\cdot b - \int_0^t\int_{\Gamma_f} p_0^{\phi_\epsilon}\cdot\mathbf{n}\cdot\boldsymbol{\psi}_\epsilon^{\phi_\epsilon}\cdot J_\epsilon,$$

$\forall\boldsymbol{\psi}_\epsilon^{\phi_\epsilon}\in\mathbf{L}^2(0,T;\mathbf{H}^1_{0,\Gamma_b\cup\Gamma_{sides}}(R)),\ b\in L^2(0,T;H_0^2(\Omega_s))$ such that

$\boldsymbol{\psi}_\epsilon^{\phi_\epsilon}(t,x_1,x_2,1) = (0,0,b(t,x_1,x_2))$ on Ω_s, div $(\mathbf{M}_\epsilon^t\boldsymbol{\psi}_\epsilon^{\phi_\epsilon}) = 0$ in R, $\boldsymbol{\psi}_\epsilon^{\phi_\epsilon}\times\mathbf{n} = 0$ on Γ_f.

The method of Galerkin

Build a basis $\{\boldsymbol{\xi}_j^0\}_{j\in\mathbf{N}}$ of the space

$$\{\mathbf{v} \in \mathbf{H}^1(R) \ : \ \text{div}\, \mathbf{v} = 0 \text{ in } R,\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_f,\ \mathbf{v} = 0 \text{ on } \partial R - \Gamma_f\}.$$

Denote $\boldsymbol{\psi}_j^0 := \mathbf{M}_\epsilon^{-t}\boldsymbol{\xi}_j^0$. The family $\{\boldsymbol{\psi}_j^0\}_{j\in\mathbf{N}}$ is a basis of the space

$$\{\mathbf{v} \in \mathbf{H}^1(R) \ : \ \text{div}\, (\mathbf{M}_\epsilon^t\mathbf{v}) = 0 \text{ in } R,\ \mathbf{M}_\epsilon^t\mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_f,\ \mathbf{v} = 0 \text{ on } \partial R - \Gamma_f\}$$

and the functions $\boldsymbol{\psi}_j^0$ are smooth in time, because $\mathbf{M}_\epsilon$ it is.

Also consider a basis $\{\rho_j\}_{j\in\mathbf{N}}$ and build functions (also smooth in time)

$\{\boldsymbol{\psi}_j^{*,\epsilon}\}_{j\in\mathbf{N}}$ s.t. $\text{div}\,(\mathbf{M}_\epsilon^t\boldsymbol{\psi}_j^{*,\epsilon}) = 0$ and $\boldsymbol{\psi}_j^{*,\epsilon}(t, x_1, x_2, 1) = (0, 0, \rho_j(x_1, x_2))$ on Ω_s.

The method of Galerkin

Looking for $\eta_\epsilon^n := \sum_{j=1}^{n} \beta_j(t)\rho_j + \eta_0^\epsilon$ and $\mathbf{u}_\epsilon^{\phi_\epsilon,m,n} := \sum_{j=1}^{m} \alpha_j(t)\boldsymbol{\psi}_j^{0,\epsilon} + \sum_{l=1}^{n} \dot{\beta}_l(t)\boldsymbol{\psi}_l^{*,\epsilon}$ such that for all $1 \leq j \leq m$,

$$\int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_j^{0,\epsilon} J_\epsilon + \nu \int_R ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}) \cdot ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \boldsymbol{\psi}_j^{0,\epsilon})$$

$$+ \int_R ((\mathbf{M}_\epsilon\nabla) \times \mathbf{v}_\epsilon^{\phi_\epsilon,\sharp}) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_j^{0,\epsilon} - \int_R (\partial_t \boldsymbol{\phi}_\epsilon \cdot (\mathbf{M}_\epsilon\nabla))\mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_j^{0,\epsilon}$$

$$-\frac{1}{2}\int_{\Gamma_{f1}} u_{\epsilon,1}^{\phi_\epsilon,m,n} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \psi_{j,1}^{0,\epsilon} J_\epsilon + \frac{1}{2}\int_{\Gamma_{f2}} u_{\epsilon,1}^{\phi_\epsilon,m,n} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \psi_{j,1}^{0,\epsilon} J_\epsilon$$

$$= \int_R \mathbf{f}^{\phi_\epsilon} \cdot \boldsymbol{\psi}_j^{0,\epsilon} J_\epsilon - \int_{\Gamma_f} p_0^{\phi_\epsilon} \cdot \mathbf{n} \cdot \boldsymbol{\psi}_j^{0,\epsilon} \cdot J_\epsilon$$

The method of Galerkin

and for all $1 \leq l \leq n$

$$\int_R \partial_t \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_l^{*,\epsilon} J_\epsilon + \nu \int_R ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}) \cdot ((\frac{\mathbf{M}_\epsilon}{\sqrt{J_\epsilon}}\nabla) \times \boldsymbol{\psi}_l^{*,\epsilon})$$

$$+ \int_R ((\mathbf{M}_\epsilon \nabla) \times \mathbf{v}_\epsilon^{\phi_\epsilon,\sharp}) \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_l^{*,\epsilon} - \int_R (\partial_t \boldsymbol{\phi}_\epsilon \cdot (\mathbf{M}_\epsilon \nabla))\mathbf{u}_\epsilon^{\phi_\epsilon,m,n} \cdot \boldsymbol{\psi}_l^{*,\epsilon}$$

$$-\frac{1}{2}\int_{\Gamma_{f1}} u_{\epsilon,1}^{\phi_\epsilon,m,n} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \psi_{l,1}^{*,\epsilon} J_\epsilon + \frac{1}{2}\int_{\Gamma_{f2}} u_{\epsilon,1}^{\phi_\epsilon,m,n} v_{\epsilon,1}^{\phi_\epsilon,\sharp} \psi_{l,1}^{*,\epsilon} J_\epsilon + \frac{1}{2}\int_{\Omega_s} \partial_t \eta_\epsilon^n \partial_t \delta_\epsilon^\sharp \rho_l + \int_{\Omega_s} \partial_{tt} \eta_\epsilon^n \rho_l$$

$$+ \int_{\Omega_s} \Delta \eta_\epsilon^n \Delta \rho_l + \gamma \int_{\Omega_s} \Delta(\partial_t \eta_\epsilon^n) \Delta \rho_l = \int_R \mathbf{f}^{\phi_\epsilon} \cdot \boldsymbol{\psi}_l^{*,\epsilon} J_\epsilon + \int_{\Omega_s} g \cdot \rho_l - \int_{\Gamma_f} p_0^{\phi_\epsilon} \cdot \mathbf{n} \cdot \boldsymbol{\psi}_l^{*,\epsilon} \cdot J_\epsilon.$$

Initial conditions:

$\beta_j(0) = 0$, $\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}(0) = \mathbf{u}_{\epsilon,0}^{\phi_\epsilon,m,n}$, where $\mathbf{u}_{\epsilon,0}^{\phi_\epsilon,m,n}$ is the projection of $\mathbf{u}_0^{\epsilon,\phi_\epsilon}$ on span $(\boldsymbol{\psi}_j^{0,\epsilon}, \boldsymbol{\psi}_l^{*,\epsilon})_{1\leq j\leq m, 1\leq l\leq n}$ and

$\partial_t \eta_\epsilon^n(0) = \eta_{01}^{\epsilon,n}$, where $\eta_{01}^{\epsilon,n}$ is the projection of η_{01}^{ϵ} on span $(\rho_l)_{1\leq l\leq n}$.

The method of Galerkin

2nd order system of ODEs $\overset{transformation}{\longrightarrow}$ 1st order ODEs system.

By the usual theory of ODEs, $\exists\, T_{m,n} > 0$ s.t. the system has a unique solution on $[0, T_{m,n}]$.

Energy estimates (independent on m, n) $\Longrightarrow$ solution on $[0, T]$:

$$||\mathbf{u}_\epsilon^{\phi_\epsilon,m,n}||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(R))} + ||\nabla \times \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(R))}$$

$$+||\partial_t \eta_\epsilon^n||_{L^\infty(0,T;H_0^2(\Omega_s))} + ||\Delta \eta_\epsilon^n||_{L^2(0,T;L^2(\Omega_s))}$$

$$\leq \text{const}\ (T, |\mathbf{u}_0||_{\mathbf{L}^2(\Omega_{\eta_0})},\ \epsilon,\ \alpha,\ ||g||_{L^2(0,T;L^2(\Omega_s))},\ ||\mathbf{f}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))},$$

$$||p_0||_{L^2(0,T;L^2(\Gamma_f))},\ ||\eta_0||_{H_0^2(\Omega_s)},\ ||\eta_{01}||_{L^2(\Omega_s)}).$$

Energy estimates (independent on $\epsilon,\ \alpha$):

$$||\mathbf{u}_\epsilon^{m,n}||_{\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega_{\delta_\epsilon^\sharp}(t)))} + ||\nabla \times \mathbf{u}_\epsilon^{m,n}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\delta_\epsilon^\sharp}(t)))} + ||\partial_t \eta_\epsilon^n||_{L^\infty(0,T;H_0^2(\Omega_s))}$$

$$+||\Delta \eta_\epsilon^n||_{L^2(0,T;L^2(\Omega_s))} \leq \text{const}\ (T, ||\mathbf{u}_0||_{\mathbf{L}^2(\Omega_{\eta_0})},\ ||g||_{L^2(0,T;L^2(\Omega_s))},$$

$$||p_0||_{L^2(0,T;L^2(\Gamma_f))},\ ||\eta_0||_{H_0^2(\Omega_s)},\ ||\eta_{01}||_{L^2(\Omega_s)},\ ||\mathbf{f}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(\mathbf{R}^3))}).$$

The method of Galerkin

Lemma 6. There exists a constant $C(\epsilon,\alpha) > 0$ such that C does not depend on m, n and

$$||\partial_t \mathbf{u}_\epsilon^{\phi_\epsilon,m,n}||_{\mathbf{L}^2(0,T;\mathbf{L}^2(R))} + ||\partial_{tt}\eta_\epsilon^n||_{L^2(0,T;L^2(\Omega_s))} \le C.$$

The above estimates allow us to pass to the limits in the linearized approximate problem.

We have: $\forall\ (\mathbf{v},\delta)$ with $\mathbf{v} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K}))$ and $\delta \in H^1(0,T;H_0^2(\Omega_s))$,

$K \ge 1 + \delta(t,x_1,x_2) \ge \alpha > 0$, $\forall\ (t,x_1,x_2) \in [0,T] \times \bar{\Omega}_s$ there exists a unique solution $(\mathbf{u}_\epsilon,\eta_\epsilon)$ of the linearized approximate problem.

Observe that $\eta_\epsilon \in H^1(0,T;H_0^2(\Omega_s))$ and that $\eta_\epsilon(0) = \eta_0^\epsilon$.

$$\tilde{\mathbf{u}}_\epsilon := \begin{cases} \mathbf{u}_\epsilon \text{ in } \Omega_{\delta_\epsilon^\sharp}(t) \\ (0,0,\partial_t\eta_\epsilon) \text{ in } \mathcal{B}_{2K} - \Omega_{\delta_\epsilon^\sharp}(t) \end{cases} \in \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K})).$$

The fixed point theorem

We define the spaces

$$\mathbf{S} := \mathbf{L}^2(0,T;\mathbf{H}^1(\mathcal{B}_{2K})) \times H^1(0,T;H_0^2(\Omega_s))$$

$$\mathbf{X} := \{(\mathbf{v},\delta) \in \mathbf{S} \ : \ ||(\mathbf{v},\delta)||_{\mathbf{S}} \leq C_X, \ \alpha \leq 1 + \delta(t,x_1,x_2) \leq K \ ,$$

$$\forall (t,x_1,x_2) \in [0,T] \times \bar{\Omega}_s, \ \delta(t=0) = \eta_0^\epsilon\}$$

and the mapping FP_ϵ:

$$\mathbf{X} \ni (\mathbf{v},\delta) \overset{FP_\epsilon}{\mapsto} (\tilde{\mathbf{u}}_\epsilon, \eta_\epsilon) \in \mathbf{S}.$$

For every $\epsilon > 0$, the mapping FP_ϵ has a fixed point. Indeed, we check the hypothesis of the Second Schauder Fixed Point Theorem:

- $\mathbf{S}$ is a reflexive, separable Banach space
- $\mathbf{X}$ is nonempty, closed, bounded and convex
- $FP_\epsilon(\mathbf{X}) \subset \mathbf{X}$
- $FP_\epsilon : \mathbf{X} \subseteq \mathbf{S} \to \mathbf{X}$ is weakly sequentially continuous

The fixed point theorem

We know that:

$(\tilde{\mathbf{u}}_\epsilon^n, \eta_\epsilon^n)_{n\in\mathbf{N}}$ is bounded in $\mathbf{S}$ and a subsequence of $(\chi_{\Omega_{\delta^\sharp_{n,\epsilon}}(t)}\partial_t \mathbf{u}_\epsilon^n, \partial_{tt}\eta_\epsilon^n)$ converges weakly in $\mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))\times L^2(0,T;L^2(\Omega_s))$.

Let $(\hat{\tilde{\mathbf{u}}}_\epsilon, \hat{\eta}_\epsilon)$ be the weak limit of $(\tilde{\mathbf{u}}_\epsilon^n, \eta_\epsilon^n)$. Show that $(\hat{\tilde{\mathbf{u}}}_\epsilon, \hat{\eta}_\epsilon) = FP_\epsilon(\mathbf{v}, \delta)$.

Pass to the limit for $n \to \infty$ in the weak formulation satisfied by $(\mathbf{u}_\epsilon^n, \eta_\epsilon^n)$:

$$\int_0^t \int_{\Omega_{\delta^\sharp_{n,\epsilon}}(s)} \partial_t \mathbf{u}_\epsilon^n \cdot \boldsymbol{\psi}_\epsilon^n + \nu \int_0^t \int_{\Omega_{\delta^\sharp_{n,\epsilon}}(s)} (\nabla\times\mathbf{u}_\epsilon^n)\cdot(\nabla\times\boldsymbol{\psi}_\epsilon^n) + \int_0^t \int_{\Omega_{\delta^\sharp_{n,\epsilon}}(s)} (\nabla\times\mathbf{v}^\sharp_{n,\epsilon})\times\mathbf{u}_\epsilon^n\cdot\boldsymbol{\psi}_\epsilon^n$$

$$-\frac{1}{2}\int_0^t\int_{\Gamma_{f1}} u^n_{\epsilon,1} v^\sharp_{n,\epsilon,1}\cdot\psi^n_{\epsilon,1} + \frac{1}{2}\int_0^t\int_{\Gamma_{f2}} u^n_{\epsilon,1} v^\sharp_{n,\epsilon,1}\cdot\psi^n_{\epsilon,1} + \frac{1}{2}\int_0^t\int_{\Omega_s} \partial_t\eta_\epsilon^n\partial_t\delta^\sharp_{n,\epsilon} b + \int_0^t\int_{\Omega_s}\partial_{tt}\eta_\epsilon^n b$$

$$+\int_0^t\int_{\Omega_s}\Delta\eta_\epsilon^n\Delta b + \gamma\int_0^t\int_{\Omega_s}\Delta(\partial_t\eta_\epsilon^n)\Delta b = \int_0^t\int_{\Omega_{\delta^\sharp_{n,\epsilon}}(s)} \mathbf{f}\cdot\boldsymbol{\psi}_\epsilon^n + \int_0^t\int_{\Omega_s} g\cdot b - \int_0^t\int_{\Gamma_f} p_0\cdot\mathbf{n}\cdot\boldsymbol{\psi}_\epsilon^n,$$

$\forall \boldsymbol{\psi}_\epsilon^n \in \mathbf{V}_{\delta^\sharp_{n,\epsilon}}$, $b \in L^2(0,T;H^2_0(\Omega_s))$ such that
$\boldsymbol{\psi}_\epsilon^n(t, x_1, x_2, 1+\delta^\sharp_{n,\epsilon}(t,x_1,x_2)) = (0,0,b(t,x_1,x_2))$ on Ω_s.

The fixed point theorem

Take $\psi_\epsilon^0 \in \mathcal{D}(Q_{\delta_\epsilon^\sharp,T})$ such that div $\psi_\epsilon^0 = 0$. Then $(\psi_\epsilon^0, 0)$ is admissible for n large enough, since $\delta^n \to \delta$ uniformly for $n \to \infty$, thus for large n's the difference between δ^n and δ is very small.

For $b \in L^2(0,T; H_0^2(\Omega_s))$ define

$$\psi_\epsilon^1(b) := \begin{cases} (0,0,b) \text{ in } \mathcal{B}_{2K} - \Omega_{\delta_\epsilon^\sharp}(t) \\ \mathbf{B}(b) \text{ in } \Omega_{\delta_\epsilon^\sharp}(t) \end{cases},$$

where $\mathbf{B}(b)$ is such that

$$\begin{aligned} \operatorname{div} \mathbf{B}(b) &= 0, \\ \mathbf{B}(b) \times \mathbf{n} &= 0 \text{ on } \Gamma_f, \\ \mathbf{B}(b) &= 0 \text{ on } \Gamma_b \cup \Gamma_{sides} \\ \mathbf{B}(b) &= (0,0,b) \text{ on } \partial\Omega_{\delta_\epsilon^\sharp}(t) - (\Gamma_f \cup \Gamma_b \cup \Gamma_{sides}) \end{aligned}$$

Then $(\psi_\epsilon^1(b), b)$ is admissible for all n.

Compactness

Let $h > 0$ be small enough; $\forall$ $\mathbf{k}(t,x)$ denote $\mathbf{k}^-(t,.) := \mathbf{k}(t-h,.)$ and $\mathbf{k}^+(t,.) := \mathbf{k}(t+h,.)$.

Lemma 7. Let $T > 0$ s.t. $\min_{[0,T]\times\bar{\Omega}_s}(1+\eta_\epsilon) \geq \alpha > 0$. Then for all $h > 0$ as above:

$$\int_0^T \int_{\mathcal{B}_{2K}} \chi_\epsilon |\tilde{\mathbf{u}}_\epsilon - \tilde{\mathbf{u}}_\epsilon^-|^2 + \int_0^T \int_{\Omega_s} (\partial_t \eta_\epsilon - \partial_t \eta_\epsilon^-)^2 \leq C h^{1/3}$$

$$\text{and} \int_0^T \int_{\mathcal{B}_{2K}} |\chi_\epsilon \tilde{\mathbf{u}}_\epsilon - \chi_\epsilon^- \tilde{\mathbf{u}}_\epsilon^-|^2 \leq C h^{1/3},$$

where for $t < 0$ we extend η_ϵ by η_0^ϵ (thus $\partial_t \eta_\epsilon$ by 0) and $\tilde{\mathbf{u}}_\epsilon$ by 0. The constant C does not depend on ϵ and χ_ϵ is the characteristic function of $\Omega_{\eta_\epsilon^\sharp}(t)$.

Frechet-Kolmogorov $\Longrightarrow$ $\chi_\epsilon \tilde{\mathbf{u}}_\epsilon$ is relatively compact in $\mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))$ and
$\partial_t \eta_\epsilon$ is relatively compact in $L^2(0,T;L^2(\Omega_s))$,
thus $\tilde{\mathbf{u}}_\epsilon$ is relatively compact in $\mathbf{L}^2(0,T;\mathbf{L}^2(\Omega_{\eta_\epsilon^\sharp}(t)))$.

Passing to the limit

Let $T > 0$ such that $\inf_\epsilon \min_{[0,T]\times\bar{\Omega}_s}(1+\eta_\epsilon) \geq \alpha > 0$; $(\widehat{\mathbf{u}}, \eta) := \lim_{\epsilon\to 0}(\tilde{\mathbf{u}}_\epsilon, \eta_\epsilon)$.

$$
\begin{aligned}
\eta_\epsilon &\overset{\epsilon\to 0}{\rightrightarrows} \eta \text{ in } C^{0,1/2}(0,T;C^{0,q}(\bar{\Omega}_s))\ (0 \leq q < 1)\\
\eta_\epsilon &\overset{\epsilon\to 0}{\rightharpoonup} \eta \text{ in } H^1(0,T;H^2(\Omega_s))\\
\partial_t\eta_\epsilon &\overset{\epsilon\to 0}{\rightarrow} \partial_t\eta \text{ in } L^2(0,T;L^2(\Omega_s))\\
\tilde{\mathbf{u}}_\epsilon &\overset{\epsilon\to 0}{\rightarrow} \widehat{\mathbf{u}} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))\\
\tilde{\mathbf{u}}_\epsilon &\overset{\epsilon\to 0}{\rightharpoonup} \widehat{\mathbf{u}} \text{ in } L^2(0,T;H^1_{0,\Gamma_b\cup\Gamma_{sides}}(\mathcal{B}_{2K}))\\
\chi_\epsilon\tilde{\mathbf{u}}_\epsilon &\overset{\epsilon\to 0}{\rightarrow} \chi\widehat{\mathbf{u}} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K}))\\
\eta_\epsilon^\sharp &\overset{\epsilon\to 0}{\rightarrow} \eta \text{ in } C^{0,1/2}(0,T;C^{0,q}(\bar{\Omega}_s))\ (0 \leq q < 1)\\
\partial_t\eta_\epsilon^\sharp &\overset{\epsilon\to 0}{\rightarrow} \partial_t\eta \text{ in } L^2(0,T;L^2(\Omega_s))\\
\tilde{\mathbf{u}}_\epsilon^\sharp &\overset{\epsilon\to 0}{\rightarrow} \widehat{\mathbf{u}} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K})),\\
\chi_\epsilon\nabla\times\mathbf{u}_\epsilon &\overset{\epsilon\to 0}{\rightharpoonup} \chi\nabla\times\widehat{\mathbf{u}} \text{ in } \mathbf{L}^2(0,T;\mathbf{L}^2(\mathcal{B}_{2K})).
\end{aligned}
$$

Passing to the limit

Preservation (for $\epsilon \to 0$) of the equality of velocities on the interface:

$$\mathbf{u}_\epsilon(t, x_1, x_2, 1 + \eta_\epsilon^\sharp(t, x_1, x_2)) = (0, 0, \partial_t \eta_\epsilon(t, x_1, x_2)) \text{ on } (0, T) \times \Omega_s.$$

Clearly, $(0, 0, \partial_t \eta_\epsilon(t, x_1, x_2)) \to (0, 0, \partial_t \eta)$ in $L^2(0, T; L^2(\Omega_s))$.
Let $(C_{\alpha/2} := \Omega_s \times (0, \alpha/2))$

$$\boldsymbol{\nu}_\epsilon := \begin{cases} (0, 0, \partial_t \eta_\epsilon) \text{ in } \mathcal{B}_{2K} - C_{\alpha/2} \\ \mathcal{R}(0, 0, \partial_t \eta_\epsilon) \text{ in } C_{\alpha/2} \end{cases},$$

where $\mathcal{R}$ is a lifting from $\mathbf{H}^{1/2}(\Omega_s \times \{\alpha/2\})$ to $\mathbf{H}^1_{0, \Gamma_b \cup \Gamma_{sides}}(C_{\alpha/2})$ such that

$$\begin{aligned} \text{div } \mathcal{R}(0, 0, \partial_t \eta_\epsilon) &= 0 \text{ in } C_{\alpha/2}, \\ \mathcal{R}(0, 0, \partial_t \eta_\epsilon)|_{(\Gamma_b \cup \Gamma_{sides}) \cap \partial C_{\alpha/2}} &= 0 \\ \mathcal{R}(0, 0, \partial_t \eta_\epsilon) \times \mathbf{n} &= 0 \text{ on } \Gamma_f \cap \partial C_{\alpha/2}. \end{aligned}$$

$\tilde{\mathbf{u}}_\epsilon - \boldsymbol{\nu}_\epsilon \in \mathbf{L}^2(0, T; \mathbf{H}^1(\mathcal{B}_{2K}))$, thus $\exists$ a subsequence $\tilde{\mathbf{u}}_\epsilon - \boldsymbol{\nu}_\epsilon \overset{\epsilon \to 0}{\rightharpoonup} \boldsymbol{\nu}_0 := \widehat{\mathbf{u}} - \boldsymbol{\nu}$ in $\mathbf{L}^2(0, T; \mathbf{H}^1(\mathcal{B}_{2K}))$ ($\boldsymbol{\nu}$ is the limit of $\boldsymbol{\nu}_\epsilon$).

$\boldsymbol{\nu}_0$ satisfies the hypotheses of Lemma 4, thus $\gamma_{\eta(t)}(\widehat{\mathbf{u}}) = (0, 0, \partial_t \eta)$.

Numerics

Approaches for the simulation of FSI problems:

- Front tracking methods:
  - The *partitioned approach* combining the Eulerian setting for the fluid with the Lagrangean formulation for the structure solves the two involved subproblems separately (with the aid of standard methods); the solution of the coupled system is obtained iteratively.
  - *Monolithical methods*: ALE (Arbitrary Lagrangean-Eulerian) method.

  However, tracking large deformations and topology changes can become very difficult when dealing with these methods.
  - In the *immersed boundary method* (Peskin '02) the idea is to get rid of any complex gridding requirements upon treating boundaries as force terms in equations satisfied in a computational domain overlapping with boundaries.

  This method is rather suited for the case where structures are low-dimensional objects immersed in the fluid.
- Front capturing methods (trying to follow the interface in an implicit way):
  - level set methods, particle methods (Cottet et al. '02, '05)
  - Eulerian/Eulerian approach (Liu & Walkington '01).

Numerics: outline of the ALE method

Notations:

- $\tilde{\Omega} := \tilde{\Omega}_f \cup \tilde{\Omega}_s$ is the initial configuration;
- $\Omega(t) := \Omega_f(t) \cup \Omega_s(t)$ is the current configuration;
- $\tilde{\mathbf{x}}$ is a point in the initial configuration;
- $\mathbf{x}$ is a point in the current configuration.

Each material particle is identified with its position in the reference configuration.

Let $\phi : \tilde{\Omega} \times (0,T) \to \Omega(t), \quad (\tilde{\mathbf{x}}, t) \mapsto \mathbf{x} = \phi(\tilde{\mathbf{x}}, t)$ satisfy $\det \nabla_{\tilde{\mathbf{x}}} \phi > 0$
(i.e., ϕ is an orientation preserving mapping).

Further notations:

- $\tilde{\mathbf{x}} \overset{\phi_t}{\mapsto} \phi(\tilde{\mathbf{x}}, t)$
- $t \overset{\phi_{\tilde{\mathbf{x}}}}{\mapsto} \phi(\tilde{\mathbf{x}}, t)$ (trajectory of the material particle $\tilde{\mathbf{x}}$)
- $\tilde{\mathbf{u}}(\tilde{\mathbf{x}}, t) = \frac{\partial \phi}{\partial t}(\tilde{\mathbf{x}}, t)$ is the velocity of the particle $\tilde{\mathbf{x}}$.

  $\mathbf{u}(\mathbf{x}, t) := \tilde{\mathbf{u}}(\tilde{\mathbf{x}}, t)$, $\mathbf{x} \in \Omega(t)$, $\mathbf{x} = \phi(\tilde{\mathbf{x}}, t)$.
- $J_\phi(\tilde{\mathbf{x}}, t) := \det \nabla_{\tilde{\mathbf{x}}} \phi(\tilde{\mathbf{x}}, t)$.

Numerics: outline of the ALE method

Since we cannot move the fluid domain $\Omega_f(t)$ along the trajectories of the material particles, we introduce another mapping, which in general does not coincide with those trajectories:

$\mathcal{A} : \tilde{\Omega}_f \times (0,T) \to \Omega_f(t)$, $(\tilde{\mathbf{x}},t) \mapsto \mathbf{x} = \mathcal{A}(\tilde{\mathbf{x}},t)$.

$\mathcal{A}$ is a computational mapping, not a physical one!

Notations:

- $\tilde{\mathbf{w}}(\tilde{\mathbf{x}},t) := \frac{\partial \mathcal{A}}{\partial t}(\tilde{\mathbf{x}},t)$ is the velocity of the fluid domain.
- For $\mathbf{x} \in \Omega(t)$ with $\mathbf{x} = \mathcal{A}(\tilde{\mathbf{x}},t)$ define $\mathbf{w}(\mathbf{x},t) := \tilde{\mathbf{w}}(\tilde{\mathbf{x}},t)$.
- The mapping $\mathcal{A}$ is s.t. $\mathcal{A} \equiv \phi$ on $\tilde{\Gamma}_{fs}$ and it is arbitrary anywhere else in $\tilde{\Omega}_f$.
- $J_{\mathcal{A}}(\tilde{\mathbf{x}},t) :=$ det $\nabla_{\tilde{\mathbf{x}}}\mathcal{A}(\tilde{\mathbf{x}},t)$.

Proposition.

$\frac{\partial J_{\mathcal{A}}}{\partial t}(\tilde{\mathbf{x}},t) = J_{\mathcal{A}}(\tilde{\mathbf{x}},t) \cdot \operatorname{div} \mathbf{w}(\mathcal{A}(\tilde{\mathbf{x}},t),t)$

$\frac{\partial J_{\phi}}{\partial t}(\tilde{\mathbf{x}},t) = J_{\phi}(\tilde{\mathbf{x}},t) \cdot \operatorname{div} \mathbf{w}(\phi(\tilde{\mathbf{x}},t),t)$.

Numerics: outline of the ALE method

Variational formulation of the FSI problem

Let $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_s$ be the Cauchy stress tensor for the fluid, respectively for the elastic structure. Then $\tilde{\boldsymbol{\Sigma}} = J_\phi \cdot \tilde{\boldsymbol{\sigma}}_s \cdot (\nabla_{\tilde{\mathbf{x}}}\phi)^{-t}$ is the *first Piola-Kirchhoff tensor*.

Let $\tilde{\mathbf{f}}_{fs} := J_\phi \cdot \tilde{\boldsymbol{\sigma}}_f \cdot (\nabla_{\tilde{\mathbf{x}}}\phi)^{-t} \cdot \tilde{\mathbf{n}}_s$ be the force exerted by the fluid on the fluid-structure interface.

FSI problem:

Fluid: Find $\mathbf{u}(t) \in \mathbf{H}^1(\Omega_f(t))$ s.t. $\mathbf{u}\Big|_{\Gamma_{fs}(t)} = \tilde{\mathbf{u}}_s\Big|_{\tilde{\Gamma}_{fs}}$ and

$$\int_{\Omega_f(t)} \frac{\partial \mathbf{u}}{\partial t}\Big|_{\tilde{\mathbf{x}}} \cdot \mathbf{v} + \int_{\Omega_f(t)} (\mathbf{u}-\mathbf{w}) \cdot \nabla \mathbf{u} \cdot \mathbf{v} + \int_{\Omega_f(t)} \boldsymbol{\sigma}_f : \nabla \mathbf{v} = 0,$$

$$\forall\, \mathbf{v} \in \mathbf{V}_f := \{\mathbf{v} \,:\, \mathbf{v}(\mathbf{x},t) = \tilde{\mathbf{v}}(\mathcal{A}_t^{-1}(\mathbf{x})),\ \tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{\Omega}_f),\ \tilde{\mathbf{v}}\Big|_{\Gamma_{fs}\cup\Gamma_f^D} = 0\}.$$

Structure: Find $\tilde{\mathbf{u}}_s(t) \in \mathbf{H}^1(\tilde{\Omega}_s)$ s.t. $\tilde{\mathbf{u}}_s\Big|_{\Gamma_s^D} = 0$ and

$$\int_{\tilde{\Omega}_s} J_\phi \frac{\partial \tilde{\mathbf{u}}_s}{\partial t} \cdot \tilde{\mathbf{v}} + \int_{\tilde{\Omega}_s} \tilde{\boldsymbol{\Sigma}} \,:\, \nabla_{\tilde{\mathbf{x}}} \tilde{\mathbf{v}} = \int_{\tilde{\Gamma}_{fs}} \tilde{\mathbf{f}}_{fs} \cdot \tilde{\mathbf{v}},$$

$$\forall\, \tilde{\mathbf{v}} \in \mathbf{V}_s := \{\tilde{\mathbf{v}} \in \mathbf{H}^1(\tilde{\Omega}_s) \,:\, \tilde{\mathbf{v}}\Big|_{\Gamma_s^D} = 0\}.$$

Velocities at the interface: Find $\mathbf{w}(t) \in \mathbf{H}^1(\Omega_f(t))$ s.t. $\mathbf{w} = \mathbf{u}$ on $\Gamma_{fs}(t)$.

Numerics: outline of the ALE method

Variational expression of the force acting on the structure:

Let $(\mathbf{u}, \boldsymbol{\sigma}_f)$ be a solution of the fluid problem above and $\mathbf{v}$ be a function defined on $\Gamma_{fs}(t)$. Let $\mathcal{R} : \Gamma_{fs}(t) \to \Omega_f(t)$ be a lifting operator.

Then

$$\int_{\tilde{\Gamma}_{fs}} \tilde{\mathbf{f}}_{fs} \cdot \tilde{\mathbf{v}} = -\int_{\Omega_f(t)} \frac{\partial \mathbf{u}}{\partial t}\Big|_{\tilde{\mathbf{x}}} \cdot \mathcal{R}(\mathbf{v}) - \int_{\Omega_f(t)} (\mathbf{u}-\mathbf{w}) \cdot \nabla \mathbf{u} \cdot \mathcal{R}(\mathbf{v}) - \int_{\Omega_f(t)} \boldsymbol{\sigma}_f : \nabla \mathcal{R}(\mathbf{v}).$$

$\frac{\partial \mathbf{u}}{\partial t}\Big|_{\tilde{\mathbf{x}}}$ denotes here and above the ALE-derivative:

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t}\Big|_{\tilde{\mathbf{x}}}(\mathbf{x}, t) &:= \frac{\partial \mathbf{u}}{\partial t}(\mathcal{A}(\tilde{\mathbf{x}}, t), t) \\
&= \frac{\partial \mathcal{A}}{\partial t}(\tilde{\mathbf{x}}, t) \cdot \nabla \mathbf{u}(\mathbf{x}, t) + \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) \\
&= \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) + \mathbf{w}(\mathbf{x}, t) \cdot \nabla \mathbf{u}(\mathbf{x}, t).
\end{aligned}$$

ALE method: discretization and numerical algorithms

Space discretization

The finite elements method (FEM) is used to discretize the variational formulation of the FSI problem.

We thereby assume that $\tilde{\Omega}_f$ and $\tilde{\Omega}_s$ are polygonal (polyhedral).

- Let $\tilde{\mathbf{V}}_h^f \subset \mathbf{H}^1(\tilde{\Omega}_f)$ be built on the mesh of $\tilde{\Omega}_f$ with e.g., P_2-finite elements.
- Let $\tilde{Q}_h^f \subset L^2(\tilde{\Omega}_f)$ be built on the mesh of $\tilde{\Omega}_f$ with e.g., P_1-finite elements.
- Let $\tilde{\mathbf{V}}_h^s \subset \mathbf{H}^1(\tilde{\Omega}_s)$ be built on the mesh of $\tilde{\Omega}_s$ with e.g., P_2-finite elements.

- Let $\mathbf{V}_h^f(\Omega_f(t)) := \{\mathbf{v}_h = \tilde{\mathbf{v}}_h \circ \mathcal{A}_t^{-1},\ \tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^f\}$.
- Let $\mathbf{V}_{h,0}^f(\Omega_f(t)) := \{\mathbf{v}_h \in \mathbf{V}_h^f(\Omega_f(t)),\ \mathbf{v}_h\Big|_{\Gamma_{fs}(t)\cup\Gamma_f^D} = 0\}$.
- Let $Q_h^f(\Omega_f(t)) := \{q_h = \tilde{q}_h \circ \mathcal{A}_t^{-1},\ \tilde{q}_h \in \tilde{Q}_h^f\}$.
- Let $\mathbf{V}_{h,0}^s(\tilde{\Omega}_s) := \{\tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^s,\ \tilde{\mathbf{v}}_h\Big|_{\Gamma_s^D} = 0\}$.

ALE method: discretization and numerical algorithms

Time discretization

- Let $\tilde{\boldsymbol{\eta}}_f$ be the displacement of the fluid domain: $\tilde{\boldsymbol{\eta}}_f = \mathcal{A}(\tilde{\mathbf{x}}, t) - \tilde{\mathbf{x}}$.
- Let $\tilde{\boldsymbol{\eta}}_s$ be the displacement of the structure: $\tilde{\boldsymbol{\eta}}_s = \phi(\tilde{\mathbf{x}}, t) - \tilde{\mathbf{x}}$.
- Let $\tilde{\boldsymbol{\eta}}_{fs}$ be the displacement of the fluid-structure interface: $\tilde{\boldsymbol{\eta}}_{fs} = \tilde{\boldsymbol{\eta}}_f\big|_{\Gamma_{fs}} = \tilde{\boldsymbol{\eta}}_s\big|_{\Gamma_{fs}}$.
- Let δt be the time step of the discretization and $t_n = n\delta t$.

Known:

- Ω_f^n (approximation of $\Omega_f(t_n)$);
- $(\mathbf{u}^n, p^n) \in \mathbf{V}_h^f(\Omega_f^n) \times Q_h^f(\Omega_f^n)$ (approximations of velocity and pressure in the corresponding discrete subspaces);
- $(\boldsymbol{\eta}_f^n, \mathbf{w}^n) \in \mathbf{V}_h^f(\Omega_f^n) \times \mathbf{V}_h^f(\Omega_f^n)$ (approximations of the fluid displacement and velocity of the fluid domain);
- $(\tilde{\boldsymbol{\eta}}_s^n, \tilde{\mathbf{u}}_s^n) \in \mathbf{V}_{h,0}^s(\tilde{\Omega}_s) \times \mathbf{V}_{h,0}^s(\tilde{\Omega}_s)$ (approximations of displacement and velocity of the structure).

Wanted: Ω_f^{n+1}, $\mathbf{u}^{n+1}$, p^{n+1}, $\tilde{\boldsymbol{\eta}}_s^{n+1}$, $\tilde{\boldsymbol{\eta}}_f^{n+1}$, $\tilde{\mathbf{u}}_s^{n+1}$.

ALE method: discretization and numerical algorithms

1. Fluid domain deformation: Assume $\tilde{\boldsymbol{\eta}}_{fs}^{n+1}$ is known.

Compute $\tilde{\boldsymbol{\eta}}_f^{n+1}$ as lifting of $\tilde{\boldsymbol{\eta}}_{fs}^{n+1}$ from $\tilde{\Gamma}_{fs}$ to $\tilde{\Omega}_f$ and define the fluid domain velocity by $\tilde{\mathbf{w}}^{n+1} = \frac{1}{\delta t}(\tilde{\boldsymbol{\eta}}_f^{n+1} - \tilde{\boldsymbol{\eta}}_f^n)$ and the new domain $\Omega_f^{n+1} := \Omega_f^n + \delta t \cdot \mathbf{w}^{n+1}$.

$$\boxed{\tilde{\boldsymbol{\eta}}_f^{n+1} = \mathcal{F}_1(\tilde{\boldsymbol{\eta}}_{fs}^{n+1})}.$$

2. Solving the fluid: Find $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{V}_h^f(\Omega_f^{n+1}) \times Q_h^f(\Omega_f^{n+1})$ s.t.

$$\begin{aligned}
\frac{1}{\delta t}\int_{\Omega_f^{n+1}} \mathbf{u}^{n+1}\cdot\mathbf{v} - \frac{1}{\delta t}\int_{\Omega_f^n} \mathbf{u}^n\cdot\mathbf{v} + 2\int_{\Omega_f^{n+1}} \nu\mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\mathbf{v}) & \\
+ \int_{\Omega_f^{n+1}} \text{div}\,(\mathbf{u}^{n+1}\otimes(\mathbf{u}^{n+1} - \mathbf{w}^{n+1}))\cdot\mathbf{v} - \int_{\Omega_f^{n+1}} p^{n+1}\text{div}\,\mathbf{v} &= 0 \\
\int_{\Omega_f^{n+1}} q\text{div}\,\mathbf{u}^{n+1} &= 0 \\
\mathbf{u}^{n+1}\Big|_{\Gamma_{fs}^{n+1}} &= \frac{1}{\delta t}(\tilde{\boldsymbol{\eta}}_{fs}^{n+1} - \tilde{\boldsymbol{\eta}}_{fs}^n).
\end{aligned}$$

$$\boxed{(\mathbf{u}^{n+1}, p^{n+1}) = \mathcal{F}_2(\tilde{\boldsymbol{\eta}}_f^{n+1})}.$$

3. Fluid force acting on the structure: Compute $\int_{\tilde{\Gamma}_{fs}} \mathbf{f}_{fs}^{n+1} \cdot \tilde{\mathbf{v}}_i$, where $\tilde{\mathbf{v}}_i\big|_{\tilde{\Gamma}_{fs}}$ is the trace on $\tilde{\Gamma}_{fs}$ of the basis function of $\tilde{\mathbf{V}}_h^s$ associated to the node i of the fluid-structure interface.

The natural choice $\int_{\Gamma_{fs}} (-p^{n+1}\mathbf{I} + 2\nu \mathbf{D}(\mathbf{u}^{n+1})) \cdot \mathbf{n} \cdot \mathbf{v}_i$ may be very difficult to compute, thus one chooses instead:

$$
\begin{aligned}
\int_{\tilde{\Gamma}_{fs}} \mathbf{f}_{fs}^{n+1} \cdot \tilde{\mathbf{v}}_i \quad = \quad & -\frac{1}{\delta t} \int_{\Omega_f^{n+1}} \mathbf{u}^{n+1} \mathcal{R}(\mathbf{v}_i\big|_{\Gamma_{fs}}) + \frac{1}{\delta t} \int_{\Omega_f^n} \mathbf{u}^n \mathcal{R}(\mathbf{v}_i\big|_{\Gamma_{fs}}) \\
& - \int_{\Omega_f^{n+1}} \mathrm{div}\, (\mathbf{u}^{n+1} \otimes (\mathbf{u}^{n+1} - \mathbf{w}^{n+1})) \cdot \mathcal{R}(\mathbf{v}_i\big|_{\Gamma_{fs}}) \\
& + \int_{\Omega_f^{n+1}} p^{n+1} \mathrm{div}\, \mathcal{R}(\mathbf{v}_i\big|_{\Gamma_{fs}}) - 2 \int_{\Omega_f^{n+1}} \nu \mathbf{D}(\mathbf{u}^{n+1}) : \mathbf{D}(\mathcal{R}(\mathbf{v}_i\big|_{\Gamma_{fs}})).
\end{aligned}
$$

$$
\boxed{\mathbf{f}_{fs}^{n+1} = \mathcal{F}_3(\mathbf{u}^{n+1}, p^{n+1})}.
$$

ALE method: discretization and numerical algorithms

4. Solving the structure:

$$\frac{1}{\delta t}\int_{\tilde{\Omega}_s} \tilde{\mathbf{u}}_s^{n+1}\cdot\tilde{\mathbf{v}} - \frac{1}{\delta t}\int_{\tilde{\Omega}_s} \tilde{\mathbf{u}}_s^{n}\cdot\tilde{\mathbf{v}} + \frac{1}{2}[a_s(\tilde{\boldsymbol{\eta}}_s^{n+1},\tilde{\mathbf{v}}) + a_s(\tilde{\boldsymbol{\eta}}_s^{n},\tilde{\mathbf{v}})] = \int_{\tilde{\Gamma}_{fs}} \mathbf{f}_{fs}^{n+1}\cdot\tilde{\mathbf{v}}\Big|_{\tilde{\Gamma}_{fs}},$$

$$\frac{1}{\delta t}\left(\tilde{\boldsymbol{\eta}}_s^{n+1} - \tilde{\boldsymbol{\eta}}_s^{n}\right) = \frac{1}{2}\left(\tilde{\mathbf{u}}_s^{n+1} + \tilde{\mathbf{u}}_s^{n}\right) \quad \text{(midpoint rule).}$$

$$\boxed{\tilde{\boldsymbol{\eta}}_{fs}^{n+1} = \mathcal{F}_4(\mathbf{f}_{fs}^{n+1})}.$$

5. Fixed-point procedure: The computation of the unknowns at time t^{n+1} requires solving the fixed-point problem

$$\boxed{\tilde{\boldsymbol{\eta}}_{fs}^{n+1} = \mathcal{F}(\tilde{\boldsymbol{\eta}}_{fs}^{n+1})},$$

with $\mathcal{F} = \mathcal{F}_4 \circ \mathcal{F}_3 \circ \mathcal{F}_2 \circ \mathcal{F}_1$.

ALE method: discretization and numerical algorithms

A fixed-point algorithm:

1. Initialization: $k = 0$, $\tilde{\boldsymbol{\eta}}_{fs,0}^{n+1} = \tilde{\boldsymbol{\eta}}_{fs}^{n}$.
2. Move the fluid domain: $\tilde{\boldsymbol{\eta}}_{f,k+1}^{n+1} = \mathcal{F}_1(\tilde{\boldsymbol{\eta}}_{fs,k}^{n+1})$.
3. Solve the fluid: $(\mathbf{u}^{n+1}, p^{n+1}) = \mathcal{F}_2(\tilde{\boldsymbol{\eta}}_{f,k+1}^{n+1})$.
4. Compute the fluid force acting on the structure: $\mathbf{f}_{fs,k+1}^{n+1} = \mathcal{F}_3(\mathbf{u}_{k+1}^{n+1}, p_{k+1}^{n+1})$.
5. Solve the structure: $\tilde{\boldsymbol{\eta}}_{fs,k+1}^{n+1} = \mathcal{F}_4(\mathbf{f}_{fs,k+1}^{n+1})$.

Until convence go to step 2.

Problem: For haemodynamics problems this algorithm quickly diverges. It needs some relaxation.

ALE method: discretization and numerical algorithms

A fixed-point algorithm with relaxation:

1. Initialization: $k = 0$, $\tilde{\boldsymbol{\eta}}_{fs,0}^{n+1} = \tilde{\boldsymbol{\eta}}_{fs}^{n} + \frac{3}{2}\delta t \tilde{\mathbf{u}}_s^n - \frac{1}{2}\delta t \tilde{\mathbf{u}}_s^{n-1}$.
2. Move the fluid domain: $\tilde{\boldsymbol{\eta}}_{f,k+1}^{n+1} = \mathcal{F}_1(\tilde{\boldsymbol{\eta}}_{fs,k}^{n+1})$.
3. Solve the fluid: $(\mathbf{u}^{n+1}, p^{n+1}) = \mathcal{F}_2(\tilde{\boldsymbol{\eta}}_{f,k+1}^{n+1})$.
4. Compute the fluid force acting on the structure: $\mathbf{f}_{fs,k+1}^{n+1} = \mathcal{F}_3(\mathbf{u}_{k+1}^{n+1}, p_{k+1}^{n+1})$.
5. Solve the structure:

$$\begin{aligned}
\widehat{\tilde{\boldsymbol{\eta}}}_{fs,k+1}^{n+1} &= \mathcal{F}_4(\mathbf{f}_{fs,k+1}^{n+1}) \\
\tilde{\boldsymbol{\eta}}_{fs,k+1}^{n+1} &= \omega_k \widehat{\tilde{\boldsymbol{\eta}}}_{fs,k+1}^{n+1} + (1-\omega_k)\tilde{\boldsymbol{\eta}}_{fs,k}^{n+1}
\end{aligned}$$

ω_k is to be chosen with the aid of the *Aitken acceleration formula* (for some references see e.g., Gerbeau '03).

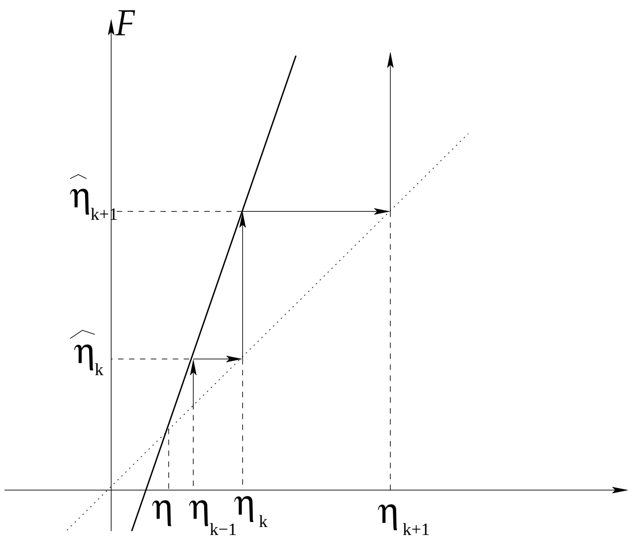
ALE method: discretization and numerical algorithms

Aitken acceleration: a method used to accelerate the convergence of the fixed point schema.

We look for a choice of ω_k. Heuristically:

1. $k = 0$, $\boldsymbol{\eta}_0$ known.
2. $$\begin{aligned} \widehat{\boldsymbol{\eta}}_{k+1} &= \mathcal{F}(\boldsymbol{\eta}_k) \\ h_{k+1} &= \widehat{\boldsymbol{\eta}}_{k+1} - \boldsymbol{\eta}_k \end{aligned}$$
3. $\boldsymbol{\eta}_{k+1} = \omega_k \widehat{\boldsymbol{\eta}}_{k+1} + (1-\omega_k)\boldsymbol{\eta}_k = \boldsymbol{\eta}_k + \omega_k h_{k+1}$.

Let us consider an intuitive, simple form of $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ (affine function)



- Observe that the algorithm diverges if the slope ω_k exceeds 1.
- How does it behave in n dimensions?

ALE method: discretization and numerical algorithms

Extension of the previous heuristic considerations to n dimensions ($\mathcal{F}$ is still 'affine'):

$\mathcal{F}\boldsymbol{\eta} = \mathbf{A}\boldsymbol{\eta} + \mathbf{b}$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$.

- We look for a fixed point $\boldsymbol{\eta}^*$, thus $\mathbf{b} = (\mathbf{I} - \mathbf{A})\boldsymbol{\eta}^*$.
- With an evaluation of $\mathcal{F}$ we can write $\widehat{\boldsymbol{\eta}}_{k+1} = \mathcal{F}(\boldsymbol{\eta}_k) = \mathbf{A}\boldsymbol{\eta}_k + \mathbf{b}$, thus $\mathbf{b} = \widehat{\boldsymbol{\eta}}_{k+1} - \mathbf{A}\boldsymbol{\eta}_k = \mathbf{h}_{k+1} + (\mathbf{I} - \mathbf{A})\boldsymbol{\eta}_k$.
- It follows that $\boldsymbol{\eta}^* = \boldsymbol{\eta}_k + (\mathbf{I} - \mathbf{A})^{-1}\mathbf{h}_{k+1}$.
- Now compare to the relaxation formula $\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + \omega_k h_{k+1}$ to find $\omega_k := \omega = (\mathbf{I} - \mathbf{A})^{-1}$.

Remarks.

- We obtain the same result with the Newton method.
- The function $\mathcal{F}$ seldom has such a nice, affine form.

ALE method: discretization and numerical algorithms

If we know two values (evaluations) of $\mathcal{F}$, than we can find the fixed point:

For the first evaluation: $\widehat{\boldsymbol{\eta}}^* = \boldsymbol{\eta}_k + \omega \mathbf{h}_{k+1}$.

For the other evaluation: $\widehat{\widehat{\boldsymbol{\eta}}}^* = \boldsymbol{\eta}_{k-1} + \omega \mathbf{h}_k$.

For a 'perfect' ω we would get $\widehat{\boldsymbol{\eta}}^* = \widehat{\widehat{\boldsymbol{\eta}}}^* = \boldsymbol{\eta}^*$.

Thus, we look for $\omega \in \mathbb{R}$ minimizing the difference between the evaluations:

$$\zeta(\omega) = \frac{1}{2}|\boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1} + \omega(\mathbf{h}_{k+1} - \mathbf{h}_k)|^2.$$

From $\zeta'(\omega) = 0$ follows then

$$\omega = \frac{1}{|\mathbf{h}_{k+1} - \mathbf{h}_k|^2}(\boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1}, \mathbf{h}_{k+1} - \mathbf{h}_k),$$

with $\mathbf{h}_{k+1} - \mathbf{h}_k = \mathcal{F}(\boldsymbol{\eta}_{k+1}) - \mathcal{F}(\boldsymbol{\eta}_k) - (\boldsymbol{\eta}_k - \boldsymbol{\eta}_{k-1})$.

Conclusions and comments

- We have pointed out some modeling aspects of FSI problems related to haemodynamics.
- We have shown the existence of a solution to the coupled steady-state problem of a 3D fluid flowing through a very long tube with thickness (periodic BCs). $\surd$ (S. '06)
- We have shown the existence of a solution to the coupled time dependent 3D fluid-3D elastic structure interaction problem in a cylindrical domain. $\surd$ (S. '07)
- We have shown the existence of a weak solution to the coupled time dependent 3D fluid-2D elastic structure interaction problem in a noncylindrical domain. $\surd$ (S. '04)
- The problem with both the cover and the bottom of the box being elastic can be treated in a similar way. $\surd$
- We outlined some popular numerical methods for simulating FSI problems.

Conclusions and comments

Better (more realistic) models:

- Existence result for a stationary 3D/3D FSI problem in a tube segment and with nonperiodic BCs. $\surd$ (S. '07)
- Existence result for the problem of a fluid contained in a cylinder with fixed ends and bounded by a thin elastic shell. $\surd$ (S. '04)
- Which are the best boundary conditions?
- What about considering longitudinal displacements?
- Permeable (nonhomogeneous) elastic walls.