PRECONDITIONING THE STAGE EQUATIONS OF IMPLICIT RUNGE KUTTA METHODS

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When using implicit Runge-Kutta methods for solving parabolic PDEs, solving the stage equations is often the computational bottleneck, as the dimension of the stage equations

$$M\mathbf{k} = \mathbf{b}$$

for an s-stage Runge-Kutta method becomes sn where the spatial discretization dimension n can be very large. Hence the solution process often requires the use of iterative solvers, whose convergence can be less than satisfactory. Moreover, due to the structure of the stage equations, the matrix M does not directly inherit any of the preferable properties of the spatial operator, making GMRES the go-to solver. Hence there is a need for a preconditioner and recently in [Neytcheva & Axelsson, 2020] and also [Howle et al., 2021 & 2022] a new block preconditioner was utilized and numerically tested with promising results.

Using spectral analysis and the particular structure of M, we study the properties of this class of preconditioners via the eigenproperties of the preconditioned systems. Concerning the eigenvalues of the preconditioned system, for low number of stages we obtain explicit formulas while in the general case we explain and cheaply calculate its characteristic features observed in [Howle et al., 2021]. As the eigenvalues alone are known to *not* be sufficient to predict the GMRES convergence behavior in general, we also focus on the eigenvectors. Combining these we get descriptive bounds of the GMRES convergence behavior for the preconditioned system.