Solving sparse-dense least squares

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Abstract

Our focus is on efficient solution of the unconstrained linear least squares (LS) problems that can be provided in the form

$$\min_x \|Ax - b\|_2, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ $(m \geq n)$ is a large sparse matrix and $b \in \mathbb{R}^m$ is given. Solving (1) is mathematically equivalent to solving the $n \times n$ *normal equations*

$$Cx = A^T b, \qquad C = A^T A. \tag{2}$$

If A has full column rank, the *normal matrix* C is symmetric and positive definite. Solving the LS problems numerically is often significantly harder than solving large and sparse systems of linear equations. One important obstacle to get efficient solvers is internal sparsity structure of the normal equations. This structure does not need to be visible when applying black-box direct methods to the solution of problem instances of small and medium size. But the structure often visibly influences solving large problems, in particular, when using preconditioned iterations. One of the implications of the encountered problems is that it is worth to develop several alternative approaches for solving different problem instances. Another implication is that a better understanding of special cases of the problem is needed. Our presentation will discuss solving the LS problems in which the system matrix contains rows with very different densities. In most situations we distinguish only two different densities: sparse and dense rows, where one assumes that the number of the dense rows is limited.

There are several classical contributions to solving the LS problem that focus on the problem, and one can find them summarized in the monograph [1] by Åke Björck and some later surveys. Such contributions typically assume a splitting of the set of input matrix rows and employ a separate processing of sparse and dense rows. In this presentation, we discuss a number of approaches. For example, we consider different ways to process sparse and dense rows of A in the preconditioner construction, specific Schur complement reductions, matrix stretching in which dense rows are replaced by submatrices with much sparser rows [7], combination of the QR factorization with the stretching [5] as well as the null-space approach [2], [6]. Experimental problems demonstrate not only strengths but also limitations of various approaches to solve these sparse-dense LS problems.

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